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Chiral Gauged WZW Theories and Coset Models in Conformal Field Theory

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The Wess-Zumino-Witten (WZW) theory has a global symmetry denoted by $G_L \otimes G_R$. In the standard gauged WZW theory, vector gauge fields (*i.e.* with vector gauge couplings) are in the adjoint representation of the subgroup $H \subset G$. In this paper, we show that, in the conformal limit in two dimensions, there is a gauged WZW theory where the gauge fields are chiral and belong to the subgroups H_L and H_R where H_L and H_R can be different groups. In the special case where $H_L = H_R$, the theory is equivalent to vector gauged WZW theory. For general groups H_L and H_R, an examination of the correlation functions (or more precisely, conformal blocks) shows that the chiral gauged WZW theory is equivalent to $(G/H)_L \otimes (G/H)_R$ coset models in conformal field theory. The equivalence of the vector gauged WZW theory and the corresponding G/H coset theory then follows.

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1. Introduction

The topological term (*i.e.* the Wess-Zumino (WZ) term[1]) in the Wess-Zumino-Witten (WZW) theory reflects the anomalies present in the theory of non-linear sigma models. Generically, the couplings of WZW actions to gauge fields contain anomalies. For external (background) gauge fields, this may be a desired feature, as in the case of the two-photon decay of neutral pions. If the gauge fields are to be treated as real degrees of freedom (*i.e.* to be integrated over in the functional integration formulation), the consistency of the resulting gauge theory requires such gauge couplings to be free from anomalies; that is, the theory must be gauge-invariant. In the absence of matter fields other than the non-linear sigma field, vector gauge couplings are automatically anomaly-free. In fact, vector gauged WZW theories have been studied extensively in the literature[2,3,4].

In this paper, we shall show that, in two dimensions (and in the conformal limit which we are mainly interested in), there exists another way to introduce gauge couplings into the WZW theory that is also anomaly-free. In this gauged WZW theory, the gauge fields are chiral, *i.e.* each gauge field has only one helicity. In this type of gauged WZW theories, the holomorphic (and anti-holomorphic) properties appear to be more transparent than in the other gauged WZW theories.

This paper is organized as follows. In Sec. 2, the coupling of the WZW actions to gauge fields and the anomaly associated with it are briefly reviewed. From the form of the non-abelian gauge anomaly, we can find special subsets of gauge couplings that are anomaly-free. Besides the vector (or equivalently the axial) gauge couplings, we show that there are two other types of gauge symmetries that are anomaly-free:

(1) local Kač−Moody (affine KM) symmetry: in this case, the local KM symmetry arises from a gauge symmetry. However, the gauge fields happen to decouple from the sigma field, leaving behind the local symmetry in the original (*i.e.* ungauged) WZW model, which is precisely the affine KM symmetry originally found by Witten[5,6,7].

(2) chiral gauge symmetry: in this case the gauge fields in the group $H_L \subset G_L$ have only the positive helicity along the left-moving light-cone ($A^L_\mu = (0, A^L_{\bar{z}})$) and the gauge fields in the group $H_R \subset G_R$ have only the negative helicity along the right-moving light-cone ($A^R_\mu = (A^R_z, 0)$).

This chiral gauged WZW theory is studied in Sec. 3. The analysis is very similar to the usual vector gauged WZW theory[3,4]. The gauge-fixed theory has separate BRST symmetries in the holomorphic and the anti-holomorphic sectors. For the left-right symmetric case, the quantized chiral gauged WZW theory is exactly equivalent to the quantized vector gauged WZW theory. However, the chiral gauged WZW theories also allow us to study the left-right asymmetric cases, which are useful for heterotic types of string theories.

In Sec. 4, we study the primary fields and their correlation functions in the gauged WZW theory. We consider these fields as the primary fields in the original (ungauged) WZW theory dressed in the "clouds" of the gauge fields. The gauge fields tend to screen the part of sigma field that belongs to the gauge group $H_{L,R}$. This picture suggests that the chiral gauged WZW theory is in fact the G/H coset theory in conformal field theory(CFT)[8,9], as was originally proposed by Schnitzer, Karabali and others for the vector gauged WZW theory[3,4]. When G and H are both simple, it will be shown that this is indeed the case by an examination of the conformal blocks. When G is semi-simple, some slight modification has to be made. To be more explicit, we will discuss in some detail this connection to coset theory for two cases: (1) $G = SU(2)_k \otimes SU(2)_l$ and $H = SU(2)$. (2) $G = SU(2)$ and $H = U(1)$. Recently it was observed that the Z_k parafermion theory[10]($i.e.$ the coset $SU(2)_k/U(1)$) may play a crucial role in the construction of string theories that have critical spacetime dimension lower than ten[11]. These new string theories have fractional supersymmetry on the world sheet. An understanding of such symmetries at the classical action level will be most useful. In fact it is the attempt to find a suitable classical action for the parafermion theory that leads to the present analysis. Also the relation of gauged WZW theories and coset models has recently received renewed interests[12,13] .

In Sec. 5, a couple of explicit examples are presented to illustrate some properties of chiral gauge theories. A number of appendices are included to make the paper self-contained. They essentially review the various derivations of the results that are needed for the main text of the paper. In Appendix A, the evaluation of the non-abelian anomaly is reviewed. Since it takes no extra effort, this derivation is presented for the arbitrary (even) dimension case. In Appendix B, the derivation of the general coupling of the WZW action to gauge fields with a given anomaly is reviewed. The evaluation of the determinant used in Sec. 3 is given in Appendix C.

2. Two Different Types of Gauged WZW Theories

In general, the coupling of the non-linear sigma field ϕ in the WZW theory to external gauge fields has non-abelian anomalies. To construct gauged WZW theories (where gauge fields are dynamical), gauge invariance must be maintained, *i.e.* the anomaly must vanish. This means only special gauge couplings can be introduced, *e.g.* vector gauge coupling. In this section, we shall show that, in two dimensions in the conformal limit, there exists another type of gauged WZW theory which is also gauge invariant, *i.e.* anomaly-free.

2.1. The Anomaly in the Effective Theory

Historically the anomaly first arose in the coupling of gauge fields to chiral fermions[14]. Let us consider the action

$$S^F = \int d^D x \left(\bar{\Psi}_R (i\partial\!\!\!/ + A\!\!\!/^R) \Psi_R + \bar{\Psi}_L (i\partial\!\!\!/ + A\!\!\!/^L) \Psi_L \right) \tag{2.1}$$

where the matrix-valued A^L_μ and A^R_μ are external gauge fields and Ψ_L and Ψ_R are multiplets of chiral fermions. Here we have suppressed all other fields and their fermionic couplings which are anomaly-free. (An example will be QCD, where vector gluons are present and couple to quarks; in this case, $A^{L,R}$ couple to flavor currents.) The effective action, $W^F_{\rm eff}$, of the above action is obtained through the path integral

$$\exp[-W^F_{\rm eff}(A^L, A^R)] \equiv \int \mathcal{D}\bar{\Psi}_R \mathcal{D}\Psi_R \mathcal{D}\bar{\Psi}_L \mathcal{D}\Psi_L \exp(-S^F) \ . \tag{2.2}$$

The infinitesimal gauge transformations are defined as

$$\begin{aligned} \delta_{\, v_L} \Psi_L(x) = v_L(x) \Psi_L(x) \ , \quad \delta_{\, v_R} \Psi_R(x) = v_R(x) \Psi_R(x) \\ \delta A^L_\mu = -\partial_\mu \ v_L + [\ v_L, A^L_\mu] \\ \delta A^R_\mu = -\partial_\mu \ v_R + [\ v_R, A^R_\mu] \end{aligned} \tag{2.3}$$

where v_L and v_R are matrix-valued and determine infinitesimally the gauge transformation. Hence the classical fermion action, S^F, is gauge invariant because of the minimal couplings in Eq.(2.1). However, the fermion integration measure may not be invariant under the gauge transformations, giving rise to the anomaly in the effective action[15]. We can express the anomaly in the following form:

$$\begin{aligned} \delta_{v_L, v_R} W^F_{\rm eff}(A^L, A^R) &\equiv W^F_{\rm eff}(A^L + \delta A^L, A^R + \delta A^R) - W^F_{\rm eff}(A^L, A^R) \\ &\equiv c_n \int d^D x \ \omega^1_{2n}(A^L, A^R; \ v_L, \ v_R) \end{aligned} \tag{2.4}$$

where the space-time dimension is $D = 2n$ and ω^1_{2n} is the non-abelian anomaly, which is a $2n$-form and linear in v_L or in v_R. The constant c_n can be determined through explicit one-loop calculations. Here $A^{L,R}$ are matrix-valued one forms, *i.e.*, $A^{L,R} \equiv A^{L,R}_{\mu,i}\lambda^i \mathbf{d}x^\mu$ where λ^i are group matrices.

The infinitesimal gauge transformation satisfies the commutation relation

$$\delta_{v_L,v_R}\delta_{v'_L,v'_R} - \delta_{v'_L,v'_R}\delta_{v_L,v_R} = \delta_{[v_L,v'_L],[v_R,v'_R]} \ . \tag{2.5}$$

This implies that the non-abelian anomaly must satisfy the following integrability condition,

$$\begin{aligned} \delta_{v_L,v_R}\int \omega^1_{2n}(A^L, A^R; v'_L, v'_R) - \delta_{v'_L,v'_R}\int \omega^1_{2n}(A^L, A^R; v_L, v_R) \\ = \int \omega^1_{2n}(A^L, A^R; [v_L, v'_L], [v_R, v'_R]) \ . \end{aligned} \tag{2.6}$$

This is the WZ consistency condition[1] and will be used later. The explicit expression for ω^1_{2n} is well-known[16]; its derivation is reviewed in Appendix A.

On the other hand, the WZW theory can be considered as a low-energy effective theory of Eq.(2.1) which must incorporate the same anomaly effect[1]. (In the QCD example mentioned above, we are interested in the effective theory of pions, where the anomaly plays a crucial role in the π^0 decay.) The WZW theory in $2n$-dimensions is formally defined by

$$S^B(\phi) = \frac{1}{4\lambda^2}\int_{S^{2n}} d^{2n}x \mathrm{Tr}\left(\partial_\mu\phi^{-1}\partial^\mu\phi\right) - C_n\int_{B^{2n+1}} \mathrm{Tr}\left(\mathbf{d}\varphi\ \varphi^{-1}\right)^{2n+1} \tag{2.7}$$

where B^{2n+1} is the $(2n+1)$-dimensional extension of the $2n$-dimensional space S^{2n}, with S^{2n} as its boundary. $\phi(x)$ is the non-linear sigma field which maps from S^{2n} to the group manifold, and $\varphi(x,t)$ is the $(2n+1)$-dimensional extension of $\phi(x)$ whose value on the boundary S^{2n} is equal to $\phi(x)$. $\mathbf{d}$ is the exterior derivative and $(\mathbf{d}\phi\phi^{-1})^{2n+1}$ is the wedge product of $(2n+1)$ one-forms. The second term is called the WZ term and will be denoted by $\Gamma(\phi)$ in the following. The effective action of the WZW theory with external gauge fields coupled to ϕ is defined as

$$\exp\left(-W^B_{\text{eff}}(A^L, A^R)\right) \equiv \int \mathcal{D}\phi \exp\left(-S^B(\phi, A^L, A^R)\right) \tag{2.8}$$

where $S^B\left(\phi, A^L, A^R\right)$ is the WZW action coupled to external gauge fields. The gauge coupling to ϕ in the first term of Eq.(2.7) can be easily incorporated via the minimal coupling, *i.e.* $\partial_\mu\phi$ is replaced by the covariant derivative $\mathrm{D}_\mu\phi$ defined as

$$\mathrm{D}_\mu\phi \equiv \partial_\mu\phi + A^L_\mu\phi - \phi A^R_\mu. \tag{2.9}$$

The term $\mathrm{Tr}\left(\mathrm{D}^{\mu}\phi^{-1}\mathrm{D}_{\mu}\phi\right)$ is invariant under the gauge transformations

$$\begin{aligned} \phi &\to g_L(x)\ \phi\ g_R^{-1}(x) \\ A_\mu^L(x) &\to g_L(x)A_\mu^L(x)g_L^{-1}(x) - \partial_\mu g_L(x)\cdot g_L^{-1} \\ A_\mu^R(x) &\to g_R(x)A_\mu^R(x)g_R^{-1}(x) - \partial_\mu g_R(x)\cdot g_R^{-1} \end{aligned} \tag{2.10}$$

under which $\mathrm{D}_\mu\phi$ behaves as $\mathrm{D}_\mu\phi \to g_L\ \mathrm{D}_\mu\phi\ g_R^{-1}$. The infinitesimal version of Eq.(2.10) is approximated by $g_{L,R}(x) \sim 1 + \ v_{L,R}(x)$. The infinitesimal transformations of $A^{L,R}$ are the same as those in Eq.(2.3) and that of the sigma field is

$$\delta\phi = \ v_L\phi - \phi\ v_R\ . \tag{2.11}$$

Under these gauge transformations, we demand the sigma field's effective action to have the same anomaly as in Eq.(2.4),

$$\begin{aligned} \delta_{v_L,v_R}W_{\text{eff}}^B(A^L,A^R) &= W_{\text{eff}}^B\left(A^L+\delta A^L, A^R+\delta A^R\right) - W_{\text{eff}}^B\left(A^L,A^R\right) \\ &= c_n\int \omega_{2n}^1(A^L,A^R;\ v_L,\ v_R)\ . \end{aligned} \tag{2.12}$$

Since the sigma field ϕ's path integral measure is gauge invariant, it follows from Eq.(2.12) that the classical action, $S^B(\phi, A^L, A^R)$, has to satisfy

$$\delta_{v_L,v_R}S^B\left(A^L,A^R,\phi\right) = c_n\int \omega_{2n}^1(A^L,A^R;\ v_L,\ v_R)\ . \tag{2.13}$$

Since the first term in S^B is gauge invariant, the anomaly only comes from the gauged WZ term.

In the gauged WZW theory, the gauge fields become dynamical gauge fields via the introduction of the kinetic term, $\mathrm{Tr}(F^2)$. In the path integral formalism, the gauge field's configurations are summed over. Since we would like to preserve the Ward identities, gauge invariance of these gauge fields must be maintained, which is equivalent to choosing a specific gauge coupling such that the anomaly ω_{2n}^1 is absent. This can be achieved in the following way. First, the anomaly $\omega_{2n}^1(A^L,A^R;\ v_L,\ v_R)$ must be determined. Next, starting from the explicit expression of ω_{2n}^1, the gauged WZW action $S^B\left(A^L,A^R,\phi\right)$ in Eq.(2.13) can be solved. Finally, among the general solutions of (2.13), we are interested in the special gauged WZW actions which are anomaly-free, *i.e.* the corresponding $\omega_{2n}^1 = 0$. It means this subset of $S^B\left(A^L,A^R,\phi\right)$ are gauge invariant.

From now on, we are interested only in the 2-dimensional gauged WZW theory in the conformal limit, where the $\mathrm{Tr}(F^2)$ term drops out. (Since $A_\mu^{L,R}$ is of dimension one, the dimension of $\mathrm{Tr}(F^2)$ is four. So the $\mathrm{Tr}(F^2)$ term drops out in the conformal limit.) The 2-dimensional WZW theory in Eq.(2.7) can be expressed as[1]

$$S(\phi) = \frac{1}{4\lambda^2} \int_{S^2} d^2x \mathrm{Tr}\left(\partial_\mu \phi^{-1} \partial^\mu \phi\right) - \frac{k}{24\pi} \int_{B^3} \mathrm{Tr}\left(\mathbf{d}\varphi\ \varphi^{-1}\right)^3 \ . \tag{2.14}$$

This action has a conformal limit, *i.e.* the infrared fixed point[5], specified by $\lambda^2 = 4\pi/k$. At this fixed point it is convenient to use the Euclidean version of the light-cone coordinate

$$z = \frac{x_1 + ix_2}{\sqrt{2}} \qquad \text{and} \qquad \bar{z} = \frac{x_1 - ix_2}{\sqrt{2}} \ . \tag{2.15}$$

The WZW action is invariant under the "local" Kač−Moody (KM) transformation denoted by $G_L(z) \otimes G_R(\bar{z})$,

$$\phi(z, \bar{z}) \rightarrow \Omega_L(z)\ \phi(z, \bar{z})\ \Omega_R^{-1}(\bar{z}) \ . \tag{2.16}$$

Note that even though this transformation is "local", no gauge fields have to be introduced to preserve the invariance of the action $S(\phi)$. This is a special property in two dimensions in the conformal limit. The symmetry generators of the above transformation are the affine KM currents, whose left currents, $J^a(z)t^a = -1/2\ k(\partial_z \phi)\phi^{-1}$, and the right ones, $\bar{J}^a(\bar{z})t^a = -1/2\ k\phi^{-1}(\partial_z \phi)$ are independent from each other[5,7]. t^a in these equations are matrices for the group representation and k is called the level in the affine KM algebra[17]. The WZW action in the conformal limit is denoted by $kI(\phi)$ in the rest of the paper.

Let us make a small digression here. It is interesting to recall that[7], for example, $G_L = G_R = SU(2)$,

$$\mathrm{Tr}\left(\partial_\mu \phi^{-1} \partial^\mu \phi\right) \sim \mathcal{G}^{(k)}(z)\ \bar{\mathcal{G}}^{(k)}(\bar{z}) \tag{2.17}$$

where $\mathcal{G}^{(k)}(z)$ is the fractional supercurrent with dimension $\frac{k+4}{k+2}$ (for $k \geq 2$). So the $SU(2)$ WZW theory can be expressed as the WZW theory in the conformal limit perturbed by the current $\mathcal{G}^{(k)}(z)$,

$$S(\phi) = kI(\phi) + \alpha(\lambda) \int d^2z\ \mathcal{G}^{(k)}(z)\ \bar{\mathcal{G}}^{(k)}(\bar{z}) \tag{2.18}$$

[1] In the following the superscript "B" or "F" which distinguishes the boson from the fermion will be neglected.

where $\alpha(\lambda)$ is a small parameter. This fractional supercurrent extends the Virasoro algebra[18] : $\mathcal{G}^{(k)}(z)$ and the stress energy-momentum tensor $T(z)$ form a non-local fractional superconformal algebra[19], which is the basis for fractional superstring[11]. (Note that for $k = 2$, this is simply the supersymmetric case.)

We have shown that the first term in Eq.(2.14) can be substituted by the gauge invariant expression, $\mathrm{Tr}\left(\mathrm{D}^{\mu}\phi^{-1}\mathrm{D}_{\mu}\phi\right)$, where the covariant derivative $\mathrm{D}_{\mu}\phi$ was defined in Eq.(2.9). Therefore, the solution to Eq.(2.13) becomes how to determine the gauge couplings in the WZ term, $\Gamma(A^L, A^R, \phi)$, *i.e.* how to solve

$$\delta_{v_L,v_R}\Gamma\left(A^L, A^R, \phi\right) = \frac{k}{8\pi}\int \omega_2^1(A^L, A^R;\ v_L,\ v_R) \tag{2.19}$$

where the constant c_2 is $k/8\pi$. However, because of the topological nature of the WZ term, the gauge couplings in $\Gamma\left(A^L, A^R, \phi\right)$ cannot be introduced simply via the minimal coupling.

Here the anomaly $\omega_2^1(A^L, A^R;\ v_L,\ v_R)$ is known[16]:

$$\omega_2^1(A^L, A^R;\ v_L,\ v_R) = \mathrm{Tr}\ (\mathbf{d}v_R\ A^R) - \mathrm{Tr}\ (\mathbf{d}v_L\ A^L)\ . \tag{2.20}$$

(See Appendix A for a derivation.) Given the non-abelian anomaly ω_2^1 above, Eq.(2.19) for the gauged WZ term can be solved explicitly, as was done in Ref.[20]. The basic idea behind the construction of $\Gamma\left(A^L, A^R, \phi\right)$ is that the WZ consistency condition (*i.e.* Eq.(2.6)) allows the integration of Eq.(2.19) along a path in the direction which can be thought of as the extra dimension needed for the extension from S^2 to B^3. The appropriate integration path should have $\Gamma(A^L, A^R, \phi)$ on the final end point, starting from an explicit function as the initial point. We review the detailed derivation in Appendix B and give the final answer here,

$$\begin{aligned}\Gamma(A^L, A^R, \phi) = &\frac{-k}{24\pi}\int_{B^3} \mathrm{Tr}\left(\varphi^{-1}\mathbf{d}\varphi\right)^3 \\ &+ \frac{k}{8\pi}\int_{S^2} \mathrm{Tr}\left(A^R\phi^{-1}\mathbf{d}\phi - \mathbf{d}\phi\phi^{-1}A^L + A^R\phi^{-1}A^L\phi\right)\ .\end{aligned} \tag{2.21}$$

This simply means that the gauged WZ term, $\Gamma(A^L, A^R, \phi)$, takes care of all the anomaly structure that we require the gauged WZW action $S(A^L, A^R, \phi)$ to satisfy, *i.e.*, Eq.(2.13).

Finally, combining Eq.(2.21) with $\mathrm{Tr}\left(\mathrm{D}_\mu\phi^{-1}\mathrm{D}^\mu\phi\right)$, we obtain the gauged WZW action:

$$
\begin{aligned}
S(A^L,A^R,\phi) &= \frac{k}{16\pi}\int d^2x\mathrm{Tr}\left(\mathrm{D}_\mu\phi^{-1}\mathrm{D}^\mu\phi\right) + \Gamma(A^L,A^R,\phi) \\
&= \frac{k}{16\pi}\int d^2x\mathrm{Tr}\left(\partial_\mu\phi^{-1}\partial^\mu\phi\right) - \frac{k}{24\pi}\int_{B^3}\mathrm{Tr}\left(\varphi^{-1}\mathbf{d}\varphi\right)^3 \\
&+ \frac{k}{8\pi}(\epsilon^{\mu\nu}+g^{\mu\nu})\int d^2x\mathrm{Tr}\left[A^R_\mu\phi^{-1}\partial_\nu\phi - \partial_\mu\phi\phi^{-1}A^L_\nu + A^R_\mu\phi^{-1}A^L_\nu\phi\right] \\
&- \frac{k}{16\pi}\int d^2x\mathrm{Tr}\left(A^{L,\mu}A^L_\mu + A^{R,\mu}A^R_\mu\right)
\end{aligned}
\tag{2.22}
$$

If we use the light-cone coordinate, $\epsilon^{z\bar{z}} = -\epsilon^{\bar{z}z} = 1 = g^{z\bar{z}} = g^{\bar{z}z}$, we can simplify the above expression as

$$
\begin{aligned}
S(A^L,A^R,\phi) =& kI(\phi) + \frac{k}{4\pi}\int d^2z\mathrm{Tr}\left[A^R_z\phi^{-1}\partial_{\bar{z}}\phi - A^L_{\bar{z}}\partial_z\phi\phi^{-1} + A^R_z\phi^{-1}A^L_{\bar{z}}\phi\right] \\
&- \frac{k}{8\pi}\int d^2z\mathrm{Tr}\left(A^L_zA^L_{\bar{z}} + A^R_zA^R_{\bar{z}}\right)
\end{aligned}
\tag{2.23}
$$

where $I(\phi)$ can be expressed in this coordinate system as

$$
I(\phi) = \frac{1}{8\pi}\int d^2z\mathrm{Tr}\left(\partial_z\phi^{-1}\partial_{\bar{z}}\phi\right) - \frac{1}{8\pi}\int_{B^3} dtd^2z\mathrm{Tr}\left(\varphi^{-1}\partial_t\varphi\varphi^{-1}\partial_z\varphi\varphi^{-1}\partial_{\bar{z}}\varphi\right). \tag{2.24}
$$

The extra dimension is denoted as t. The gauge variation of $S(A^L,A^R,\phi)$ is given by Eq.(2.13):

$$
\delta_{v_L,v_R}S(A^L,A^R,\phi) = \frac{k}{8\pi}\int d^2z\mathrm{Tr}\left(\partial_z v_R A^R_{\bar{z}} - \partial_{\bar{z}}v_R A^R_z - \partial_z v_L A^L_{\bar{z}} + \partial_{\bar{z}}v_L A^L_z\right) \tag{2.25}
$$

Based on Eq.(2.25) we can discuss at least three possible scenarios in which the action $S(A^L,A^R,\phi)$ given in Eq.(2.23) is invariant under the gauge transformation, *i.e.* the non-abelian anomaly on the R.H.S. of Eq.(2.25) vanishes.

(I.) Vector Gauge Invariance:

The vector gauge invariance can be obtained if we choose the gauge fields and the gauge transformation parameters to be

$$
\begin{aligned}
A^{\mathrm{vec}}_z(z,\bar{z}) = A^R_z(z,\bar{z}) \equiv& A^L_z(z,\bar{z})\ , \qquad A^{\mathrm{vec}}_{\bar{z}}(z,\bar{z}) = A^R_{\bar{z}}(z,\bar{z}) \equiv A^L_{\bar{z}}(z,\bar{z}) \\
&v(z,\bar{z}) = v_R(z,\bar{z}) \equiv v_L(z,\bar{z})\ .
\end{aligned}
\tag{2.26}
$$

Thus, Eq.(2.23), can be written as

$$
\begin{aligned}
S(A_z^{\text{vec}}, A_{\bar z}^{\text{vec}}, \phi) = kI(\phi) + \frac{k}{4\pi}\int d^2z \text{Tr}\Big[&A_z^{\text{vec}}\phi^{-1}\partial_{\bar z}\phi - A_{\bar z}^{\text{vec}}\partial_z\phi\phi^{-1} \\
&+ A_z^{\text{vec}}\phi^{-1}A_{\bar z}^{\text{vec}}\phi - A_z^{\text{vec}}A_{\bar z}^{\text{vec}}\Big]
\end{aligned}
\tag{2.27}
$$

The vector gauge invariance can be verified from Eq.(2.25):

$$
\delta_{v,v}S(A_z^{\text{vec}}, A_{\bar z}^{\text{vec}}, \phi) = 0 \ . \tag{2.28}
$$

This vector gauged action $S(A_z^{\text{vec}}, A_{\bar z}^{\text{vec}}, \phi)$ has been extensively studied in the literature[3,4,12]. We note that the axial gauge coupling can be introduced instead of the vector gauge coupling. In that case

$$
\begin{aligned}
A_z^{\text{a}}(z,\bar z) = A_z^L(z,\bar z) \equiv -\, A_z^R(z,\bar z)\ , \qquad A_{\bar z}^{\text{a}}(z,\bar z) = A_{\bar z}^L(z,\bar z) \equiv -A_{\bar z}^R(z,\bar z) \\
v(z,\bar z) = v_L(z,\bar z) \equiv -v_R(z,\bar z)\ .
\end{aligned}
\tag{2.29}
$$

(II.) Local Kač-Moody Invariance:

It is instructive to see how the local KM symmetry, Eq.(2.16), emerges from our formalism. Let us choose

$$
\begin{aligned}
A_{\bar z}^L \equiv A_z^R \equiv 0 \\
v_L \equiv v_L(z) \qquad \text{and} \quad v_R \equiv v_R(\bar z)
\end{aligned}
\tag{2.30}
$$

so that the gauge fields become

$$
A_\mu^L = (A_z^L(z,\bar z), 0) \quad \text{and} \quad A_\mu^R = (0, A_{\bar z}^R(z,\bar z))\ . \tag{2.31}
$$

In this case, the gauge fields, A_z^L and $A_{\bar z}^R$ decouple from the ϕ field in $S(A_z^L, A_{\bar z}^R, \phi)$, *i.e.* $S(A_z^L, A_{\bar z}^R, \phi) = kI(\phi)$. Therefore we recover the original WZW action. Under the gauge transformation

$$
\delta_{v_L(z),v_R(\bar z)}S(A_z^L, A_{\bar z}^R, \phi) = \delta_{v_L(z),v_R(\bar z)}\ kI(\phi) = 0 \tag{2.32}
$$

This gauge symmetry is simply the infinitesimal version of the local transformation in Eq.(2.16), *i.e.* $\phi(z,\bar z) \to \Omega_L(z)\phi(z,\bar z)\Omega_R^{-1}(\bar z)$. Usually the local gauge symmetry implies the introduction of gauge fields. Hence the local symmetry, Eq.(2.16), of $kI(\phi)$ without gauge fields is somewhat mysterious. From our point of view, there are gauge fields A_z^L, $A_{\bar z}^R$

associated with this local KM symmetry. It just so happens that these gauge fields simply decouple from the action $S(A^L_z, A^R_{\bar z}, \phi)$ in Eq.(2.23).

(III.) Chiral Gauge Invariance:

Let us consider another case:

$$\begin{aligned} &A^L_z \equiv A^R_{\bar z} \equiv 0 \\ &v_L = v_L(\bar z) \qquad \text{and} \quad v_R = v_R(z) \end{aligned} \tag{2.33}$$

so that the gauge fields become

$$A^L_\mu = (0, A^L_{\bar z}(z,\bar z)) \quad \text{and} \quad A^R_\mu = (A^R_z(z,\bar z), 0)\ . \tag{2.34}$$

Thus, Eq.(2.23) becomes

$$S(A^L_{\bar z}, A^R_z, \phi) = kI(\phi) + \frac{k}{4\pi}\int d^2z \mathrm{Tr}\left[A^R_z\phi^{-1}\partial_{\bar z}\phi - A^L_{\bar z}\partial_z\phi\phi^{-1} + A^R_z\phi^{-1}A^L_{\bar z}\phi\right] \tag{2.35}$$

which will be denoted as the chiral gauged WZW action. It has the gauge symmetry following from Eq.(2.25)

$$\delta_{v_L(\bar z), v_R(z)} S(A^L_{\bar z}, A^R_z, \phi) = 0 \tag{2.36}$$

This gauge symmetry is new and exists only in 2-dimensions. It has the advantage that the left gauge degrees of freedom are separated from the right ones, *i.e.* $v_L \neq v_R$. Both the gauge fields, left-handed and right-handed, have only one component, $A^L_{\bar z}(z,\bar z)$ and $A^R_z(z,\bar z)$ respectively. In general the chiral gauge invariant action, Eq.(2.35) is different from Eq.(2.27). The quantization of this chiral gauged WZW theory will be explored in the next section.

It is worthwhile to mention that the quantization of the gauge fields with the vector gauge invariant action, Eq.(2.27) has been studied extensively in the literature[3,4]. However, we shall see that the connection between the coset theory in the CFT and the gauged WZW theory has a natural setting in terms of the chiral gauged WZW theory because it has explicit left-right independent gauge fields. In Table 1 we summarize the gauge symmetries of the two different types of gauged WZW models discussed above.

	Chiral Gauge Theory	Vector Gauge Theory
Action	Eq.(2.35)	Eq.(2.27)
Gauge fields	$A^L_\mu = (0\ ,\ A^L_{\bar z}(z,\bar z))$	$A^{\rm vec}_\mu = (A^{\rm vec}_z(z,\bar z)\ ,\ A^{\rm vec}_{\bar z}(z,\bar z))$
	$A^R_\mu = (A^R_z(z,\bar z)\ ,0)$	
Gauge symmetry	$\delta\phi = v_L(\bar z)\ \phi - \phi\ v_R(z)$	$\delta\phi = v(z,\bar z)\ \phi - \phi\ v(z,\bar z)$
	$\delta A^L_{\bar z} = -\partial_{\bar z} v_L(\bar z) + [v_L,\ A^L_{\bar z}]$	$\delta A^{\rm vec}_{\bar z} = -\partial_{\bar z} v(z,\bar z) + [v,\ A^{\rm vec}_{\bar z}]$
	$\delta A^R_z = -\partial_z v_R(z) + [v_R,\ A^R_z]$	$\delta A^{\rm vec}_z = -\partial_z v(z,\bar z) + [v,\ A^{\rm vec}_z]$

Table 1. Chiral and vector gauged WZW theories.

It will be shown in Sec. 3 that when the left gauge group is the same as the right one in the chiral gauge theory, *i.e.* $H_L = H_R$, the quantized chiral gauge theory is the same as the quantized vector gauge theory.

3. Quantizations of the Chiral Gauged WZW Theory

The quantization procedure for the chiral gauged WZW theory is very similar to that for the vector gauged WZW theory[3]. Here we shall give a self-contained presentation.

Formally the partition function is defined by

$$\mathcal{Z} = \int \mathcal{D}A^R_z \mathcal{D}A^L_{\bar z} \mathcal{D}\phi \exp\left(-S(A^L_{\bar z}, A^R_z, \phi)\right) \tag{3.1}$$

where $S(A^L_{\bar z}, A^R_z, \phi)$ is given by Eq.(2.35). The gauge fields $A^L_{\bar z}$ and A^R_z belong to the adjoint representations of gauge groups H_L and H_R which are subgroups of G_L and G_R respectively. $A^L_{\bar z}(z,\bar z)$ and $A^R_z(z,\bar z)$ can be parametrized in terms of $h(z,\bar z)$ and $\widetilde{h}(z,\bar z)$ by

$$A^L_{\bar z} = h^{-1}\partial_{\bar z} h \qquad \text{and} \qquad A^R_z = -\partial_z \widetilde{h}\widetilde{h}^{-1} \tag{3.2}$$

These parametrizations simplify the action $S(A^L, A^R, \phi)$ and facilitate the separation of the gauge degrees of freedom from the physical degrees of freedom. By exploiting the Polyakov-Wiegmann formula[6]

$$I(g\phi) = I(g) + I(\phi) - \frac{1}{4\pi}\int d^2x \mathrm{Tr}\left(g^{-1}\partial_{\bar z} g\ \partial_z\phi\cdot\phi^{-1}\right)\ , \tag{3.3}$$

$S(A^L_{\bar z}, A^R_z, \phi)$ can be rewritten in terms of $h, \widetilde{h}$ and $\widehat{\phi} \equiv h\phi\widetilde{h}$, *i.e.* $S(A^L_{\bar z}, A^R_z, \phi) = S(h, \widetilde{h}, \widehat{\phi})$ where

$$S(h, \widetilde{h}, \widehat{\phi}) = kI(\widehat{\phi}) - kI(h) - kI(\widetilde{h}) \tag{3.4}$$

The gauge transformations on h, $\widetilde{h}$ and $\widehat{\phi}$ can be found from the transformations on $A^L_{\bar z}$, A^R_z and ϕ,

$$
\begin{aligned}
A^L_{\bar z} &\to g_L(\bar z) A^L_{\bar z} g_L^{-1}(\bar z) - \partial_{\bar z} g_L(\bar z)\cdot g_L^{-1}(\bar z) = (h\ g_L^{-1})^{-1}\partial_{\bar z}(h\ g_L^{-1})\\
&\quad \Rightarrow h \to h' \equiv h\cdot g_L^{-1}(\bar z)\\
A^R_z &\to g_R(z) A^R_z g_R^{-1}(z) - \partial_z g_R(z)\cdot g_R^{-1}(z) = -\partial_z(g_R\ \widetilde{h})\cdot(g_R\ \widetilde{h})^{-1}\\
&\quad \Rightarrow \widetilde{h} \to \widetilde{h}' \equiv g_R(z)\cdot\widetilde{h}\\
\phi &\to g_L(\bar z)\ \phi\ g_R^{-1}(z) \qquad\qquad \Rightarrow \widehat{\phi}\to\widehat{\phi}\ ,
\end{aligned}
\tag{3.5}
$$

where $g_L(\bar z)$ and $g_R(z)$ are the finite gauge transformations determined by $v_L(\bar z)$ and $v_R(z)$ respectively. From Eq.(3.2), we see that $A^L_{\bar z}$ is invariant under the transformation $h(z,\bar z) \to U_L(z)h(z,\bar z)$. Similarly, A^R_z is invariant under the transformation $\widetilde{h}(z,\bar z) \to \widetilde{h}(z,\bar z)U_R(\bar z)$. These are half of the affine KM symmetry. Hence the action Eq.(3.4) is invariant under

$$
\begin{aligned}
&h \to U_L(z)\ h\ g_L^{-1}(\bar z), \qquad \widetilde{h}\to g_R(z)\ \widetilde{h}\ U_R(\bar z)\\
&\widehat{\phi} = h\ \phi\ \widetilde{h} \to U_L(z)\ \widehat{\phi}\ U_R(\bar z)
\end{aligned}
\tag{3.6}
$$

Following standard procedures, we shall remove the gauge volume in $\mathcal{Z}$ in Eq.(3.1). In the new variables $(h, \widetilde{h}, \widehat{\phi})$, we must also factor out the half-KM symmetry volume (due to the symmetries, $U_L(z)$ and $U_R(\bar z)$) in the partition function. This can be achieved by picking out particular $h_{\rm p}(z,\bar z)$ and $\widetilde{h}_{\rm p}(z,\bar z)$ from their respective gauge orbits and "half-KM orbits". We shall refer to this procedure loosely as gauge-fixing.

The Jacobian for the change of variable Eq.(3.2) can be calculated from

$$
\mathcal{D}A^L_{\bar z}\mathcal{D}A^R_z\mathcal{D}\phi = J\mathcal{D}h\ \mathcal{D}\widetilde{h}\ \mathcal{D}\widehat{\phi}
\tag{3.7}
$$

where J is the determinant of an upper triangle matrix. The variation of $A^{L,R}$ can be determined by that of h, $\widetilde{h}$,

$$
\begin{aligned}
\delta A^L_{\bar z} &= \delta(h^{-1})\partial_{\bar z} h + h^{-1}\partial_{\bar z}(\delta h) \equiv {\rm D}^L_{\bar z}(\delta h^{-1}\cdot h)\\
\delta A^R_z &= -\partial_z(\delta\widetilde{h})\widetilde{h}^{-1} - \partial_z\widetilde{h}\ \delta(\widetilde{h}^{-1}) \equiv {\rm D}^R_z(\delta\widetilde{h}\cdot\widetilde{h}^{-1})
\end{aligned}
\tag{3.8}
$$

where ${\rm D}^{L,R}_\mu$ is the covariant derivative, ${\rm D}^{L,R}_\mu \equiv -\partial_\mu - [A^{L,R}_\mu,\quad]$. Therefore the Jacobian is

$$
J = det({\rm D}^L_{\bar z})\ det({\rm D}^R_z)\ .
\tag{3.9}
$$

These two determinants are evaluated in Appendix C and are equal to

$$
\begin{aligned}
\left(det\ \mathrm{D}_{\bar{z}}^{L}\right) &= \exp\left[2C_2(H_L)I(h)\right]\cdot\left(\prod_{\alpha,\beta=1}^{dim\ H_L} det\ \delta^{\alpha\beta}\ \partial_z\right) \\
\left(det\ \mathrm{D}_{\bar{z}}^{R}\right) &= \exp\left[2C_2(H_R)I(\widetilde{h})\right]\cdot\left(\prod_{\alpha,\beta=1}^{dim\ H_R} det\ \delta^{\alpha\beta}\ \partial_{\bar{z}}\right)\ .
\end{aligned}
\tag{3.10}
$$

where C_2 is the dual Coxeter number, defined by the structure constant of the gauge group, $\sum_{\beta\gamma} f^{a\beta\gamma}f^{\alpha\beta\gamma} = -C_2\delta^{a\alpha}$.

Finally we can combine Eq.(3.1), (3.2), (3.4), (3.9), (3.10) and exponentiate the determinants $\left(det\ \delta^{\alpha\beta}\ \partial_{\bar{z}}\right)$ and $\left(det\ \delta^{\alpha\beta}\ \partial_z\right)$ to rewrite the full quantum partition function as

$$
\begin{aligned}
\mathcal{Z}_{\mathrm{gauged-fixed}} &= \int \mathcal{D}h_{\mathrm{p}}\mathcal{D}\widetilde{h}_{\mathrm{p}}\mathcal{D}\widehat{\phi}\exp\left(-kI(\widehat{\phi})\right) \\
&\left[\exp\Big(\left[k+2C_2(H_L)\right]I(h_{\mathrm{p}})\Big)\left(\prod_{\alpha=1}^{dimH_L}\mathcal{D}b_z^{\alpha}\mathcal{D}c^{\alpha}\exp\left(-\int d^2z b_z^{\alpha}\bar{\partial}c^{\alpha}\right)\right)\right] \\
&\left[\exp\left(\left[k+2C_2(H_R)\right]I(\widetilde{h}_{\mathrm{p}})\right)\left(\prod_{\alpha=1}^{dimH_R}\mathcal{D}\bar{b}_{\bar{z}}^{\alpha}\mathcal{D}\bar{c}^{\alpha}\exp\left(-\int d^2z\bar{b}_{\bar{z}}^{\alpha}\partial\bar{c}^{\alpha}\right)\right)\right]
\end{aligned}
\tag{3.11}
$$

where b_z^{α}, c^{α} (and $\bar{b}_{\bar{z}}^{\alpha}, \bar{c}^{\alpha}$) are spin $(1,0)$ ghosts introduced for each generator of the gauge group H_L (and H_R). The 2-point function, mode expansions and anticommutations of b_z^{α}, c^{α} are standard:

$$
\begin{aligned}
b_z^{\alpha}(z)c^{\beta}(w) &= \frac{\delta^{\alpha\beta}}{z-w} \quad \text{and} \quad b_z^{\alpha}(z)b_z^{\beta}(w) = c^{\alpha}(z)c^{\beta}(w) = 0 \\
b_z^{\alpha}(z) &= \sum_n z^{-n-1}b_{z,n}^{\alpha} \quad \text{and} \quad c^{\alpha}(z) = \sum_n z^{-n}c_n^{\alpha} \\
\{b_{z,n}^{\alpha}\ ,\ \ c_m^{\beta}\} &= \delta^{\alpha\beta}\delta_{n+m} \quad \text{and} \quad \{b_{z,n}^{\alpha}\ ,\ \ b_{z,m}^{\beta}\} = \{c_n^{\alpha}\ ,\ \ c_m^{\beta}\} = 0\ .
\end{aligned}
\tag{3.12}
$$

$\bar{b}_{\bar{z}}^{\alpha}, \bar{c}^{\alpha}$ have similar constructions.

As familiar in the quantum gauge theory, the inclusion of the ghost fields allows us to construct the BRST operator whose nilpotency guarantees that the gauge symmetry is maintained at the quantum level. The construction of the BRST operators relies on the remaining symmetries in the gauge-fixed $\mathcal{Z}$ in Eq.(3.11). To find out such symmetry we first review the affine KM symmetries in the WZW actions $I(h_{\mathrm{p}})$, $I(\widetilde{h}_{\mathrm{p}})$ and $I(\widetilde{\phi})$. The KM currents are denoted by $J_h^{\alpha}(z)$, $\bar{J}_{\tilde{h}}^{\beta}(\bar{z})$, $J_{\phi}^{a}(z)$ and $\bar{J}_{\phi}^{a}(\bar{z})$ where $a = 1, 2, ..., dim\ G$,

$\alpha = 1, 2, ..., dim\ H_L$ and $\beta = 1, 2, ..., dim\ H_R$. Those currents have the operator product expansions(OPEs)

$$
\begin{aligned}
J^a_\phi(z) J^b_\phi(w) &= \frac{\delta^{ab} k/2}{(z-w)^2} + \frac{f^{abc} J^c_\phi}{(z-w)} + \text{regular terms.} \\
J^\alpha_h(z) J^\beta_h(w) &= \frac{\delta^{\alpha\beta}[-k/2 - C_2(H_L)]}{(z-w)^2} + \frac{f^{\alpha\beta\gamma} J^\gamma_h}{(z-w)} + \text{regular terms.}
\end{aligned}
\tag{3.13}
$$

where f^{abc} and $f^{\alpha\beta\gamma}$ are the structure constants of the group G_L and the gauge group H_L respectively. The right-handed currents can be similarly constructed. The mode expansions in terms of moding operators of these currents are $J^a_\phi(z) = \sum_n z^{n-1} J^a_{\phi,-n}$ and $\bar{J}^a_\phi$, J^α_h and $\bar{J}^\alpha_{\bar{h}}$ have similar expressions. The commutation relations between the moding operators can be obtained through the above OPEs. The ghost action in Eq.(3.11) also has affine KM symmetries generated by KM ghost currents

$$
J^\alpha_{\text{gh}}(z) = -f^{\alpha\beta\gamma} : b^\beta_z c^\gamma : (z) \tag{3.14}
$$

which satisfy affine KM algebra of group H_L at level $2C_2(H_L)$. So we can define

$$
\begin{aligned}
J^a_{\text{tot}}(z) &\equiv J^a_\phi(z) \qquad \text{for} \ \ a \in G_L, \ \ \text{and} \ \ a \notin H_L \\
J^\alpha_{\text{tot}}(z) &\equiv J^\alpha_\phi(z) + J^\alpha_h(z) + J^\alpha_{\text{gh}}(z) \qquad \text{for} \ \ \alpha \in H_L \ .
\end{aligned}
\tag{3.15}
$$

Note that the $(z-w)^{-2}$ term is absent in the OPE

$$
J^\alpha_{\text{tot}}(z) J^\beta_{\text{tot}}(w) = \frac{f^{\alpha\beta\gamma} J^\gamma_{\text{tot}}(w)}{z-w} + ... \qquad \text{for} \ \ \alpha, \beta \ \in H_L \tag{3.16}
$$

i.e. the $\{J^\alpha_{\text{tot}}(z), \ \ \text{for} \ \alpha \in H_L\}$ current algebra is free from affine KM anomaly as required by the gauge invariance. Similarly the antiholomorphic KM currents $\bar{J}^\alpha_{\text{tot}}(\bar{z})$ can be constructed. Hence the holomorphic and antiholomorphic BRST operators can be found

$$
\begin{aligned}
Q^L_{\text{BRST}} &= \oint dz \prod_{\alpha=1}^{dim H_L} c^\alpha(z) J^\alpha_{\text{tot}}(z) \\
Q^R_{\text{BRST}} &= \oint d\bar{z} \prod_{\alpha=1}^{dim H_R} \bar{c}^\alpha(\bar{z}) \bar{J}^\alpha_{\text{tot}}(\bar{z})
\end{aligned}
\tag{3.17}
$$

These two BRST operators correspond exactly to the gauge symmetry generated by $v_L(\bar{z})$ and $v_R(z)$. They can also be written in terms of the moding operators of the currents and the b, c ghost, *e.g.*

$$
Q^L_{BRST} = \sum_{\infty}^{\infty} c^\alpha_{-n} \left(J^\alpha_{\phi,n} + J^\alpha_{h,n} \right) - \frac{1}{2} f^{\alpha\beta\gamma} : c^\alpha_{-n} b^\beta_{-m} c^\gamma_{n+m} : \tag{3.18}
$$

The anticommutation relations between these BRST operators can be easily checked from Eq.(3.12), (3.13) and (3.17),

$$\left\{Q^L_{BRST}\ ,\ \ Q^L_{BRST}\right\} = \left\{Q^R_{BRST}\ ,\ \ Q^R_{BRST}\right\} = \left\{Q^L_{BRST}\ ,\ \ Q^R_{BRST}\right\} = 0 \tag{3.19}$$

which guarantee the nilpotency of the BRST operators.

There is a special case in which h_p and $\widetilde{h}_\mathrm{p}$ fields decouple from the matter and ghost system. It can be seen from Eq.(3.11) that when the level $k = -2C_2(H_L)$, the h_p field decouples from the action and the integration over h_p becomes an irrelevant volume factor which can be factored out from the path integral. This can also be seen from the construction of the BRST operators without including h_p in Q^L_{BRST}, *i.e.*

$$Q^L_{BRST} = \oint dz \prod_{\alpha=1}^{dim H_L} c^\alpha(z)\left[J^\alpha_\phi(z) + J^\alpha_{\mathrm{gh}}(z)\right] \tag{3.20}$$

which satisfies

$$\left\{Q^L_{BRST}\ ,\ \ Q^L_{BRST}\right\} = (k + 2C_2(H_L))\sum_{m=1}^{\infty} m c^\alpha_{-m} c^\alpha_m. \tag{3.21}$$

Hence the nilpotency of Q^L_{BRST} without h_p is guaranteed if the level $k = -2C_2(H_L)$; similarly Q^R_{BRST} is nilpotent without $\widetilde{h}_\mathrm{p}$ if the level $k = -2C_2(H_R)$. Therefore, k has the *critical* value when $k = -2C_2(H_L) = -2C_2(H_R)$ because h_p and $\widetilde{h}_\mathrm{p}$ both decouple from the theory. Thus the chiral gauged WZW theory is remarkably similar to the vector gauged WZW theory. The Q_{BRST} in Eq.(3.17) was first considered in Ref.[3] and the special case Eq.(3.20) was first considered in Ref.[21].

4. Chiral Gauged WZW Theories and Coset Models

Now we are ready to demonstrate the equivalence of the gauged WZW theories and the coset models in CFT. Our analysis is motivated by the work on the two-dimensional quantum gravity[22] and the starting point is Knizhnik and Zamolodchikov's formulation of WZW theory. The primary fields in the chiral gauged WZW theory are considered as the primary fields in the original (*i.e.* ungauged) WZW theory dressed in the "clouds" of the gauge fields. By an examination of the correlation functions (or more precisely the conformal blocks) of these dressed primary fields, we shall show that the chiral gauged WZW theory are isomorphic to the coset theory G/H. This result follows from the fact that the "gauge clouds" surrounding the primary fields of the original ungauged WZW theory are precisely the primary fields of H but with a time-like metric. So they effectively cancel that part of the primary fields of G which belongs to H. For the sake of concreteness, we will consider two examples in some detail: (1) $G = SU(2)_k$ with $H_L = H_R = U(1)$ and its equivalence to the Z_k parafermion (*i.e.* $SU(2)_k/U(1)$ coset) theory; (2) $G = SU(2)_k \otimes SU(2)_l$ with $H = SU(2)$ will be shown to be isomorphic to the coset theory $SU(2)_k \otimes SU(2)_l/SU(2)_{k+l}$.

As discussed in the last section, there are affine Kač−Moody currents $J^a_\phi(z)$ and $\bar{J}^a_\phi(\bar{z})$ in the WZW theory, whose OPEs are listed in Eq.(3.13). The primary fields $G^{\Lambda,\bar{\Lambda}}_{\lambda,\bar{\lambda}}(z,\bar{z})$ of the WZW theory with quantum number $\lambda, \bar{\lambda}$ in the representations $\Lambda, \bar{\Lambda}$ satisfy

$$
\begin{aligned}
J^a_\phi(z) G^{\Lambda,\bar{\Lambda}}_{\lambda,\bar{\lambda}}(w,\bar{w}) &= \frac{(t^a_\Lambda)_{\lambda,\lambda'} G^{\Lambda,\bar{\Lambda}}_{\lambda',\bar{\lambda}}(w,\bar{w})}{z-w} + \text{reg. terms} \\
\bar{J}^a_\phi(\bar{z}) G^{\Lambda,\bar{\Lambda}}_{\lambda,\bar{\lambda}}(w,\bar{w}) &= \frac{(t^a_{\bar{\Lambda}})_{\bar{\lambda},\bar{\lambda}'} G^{\Lambda,\bar{\Lambda}}_{\lambda,\bar{\lambda}'}(w,\bar{w})}{\bar{z}-\bar{w}} + \text{reg. terms}
\end{aligned}
\tag{4.1}
$$

where t^a_Λ and $t^a_{\bar{\Lambda}}$ are the matrices of the group generators in the representations Λ and $\bar{\Lambda}$ respectively. The Fock space of the WZW theory is built up by applying the negative modings of the holomorphic and antiholomorphic currents on $G^{\Lambda,\bar{\Lambda}}_{\lambda,\bar{\lambda}}(w,\bar{w})$. Because $J^a_\phi(z)$'s are independent from $\bar{J}^a_\phi(\bar{z})$'s, we can separate the holomorphic and antiholomorphic parts of the fields in the WZW theory, *i.e.* $G^{\Lambda,\bar{\Lambda}}_{\lambda,\bar{\lambda}}(w,\bar{w}) = G^{\Lambda}_{\lambda}(w) G^{\bar{\Lambda}}_{\bar{\lambda}}(\bar{w})$ and similarly for all other states in the Fock space. The holomorphic part of the stress energy-momentum tensor in the WZW theory is given by the Sugawara form

$$
T_\phi(z) = \frac{1}{k + C_2(G)} \sum_{a=1}^{dim\ G} : J^a_\phi(z) J^a_\phi(z) : \ . \tag{4.2}
$$

Therefore the highest weights of $G^{\Lambda,\bar{\Lambda}}_{\lambda,\bar{\lambda}}(w,\bar{w})$ can be obtained from Eq.(4.2) and (4.1)

$$
\Delta^\Lambda_G = \frac{C_\Lambda(G)}{k + C_2(G)} \quad \text{and} \quad \bar{\Delta}^{\bar{\Lambda}}_G = \frac{C_{\bar{\Lambda}}(G)}{k + C_2(G)} \tag{4.3}
$$

where C_Λ is the quadratic Casimir of the representation Λ.

In the chiral gauged WZW theory, the currents are given in Eq.(3.15),

$$
\begin{aligned}
J^a_{\text{tot}}(z) &\equiv J^a_\phi(z) \qquad \text{for} \ \ a \in G_L, \ \ \text{and} \ \ a \notin H_L \\
J^\alpha_{\text{tot}}(z) &\equiv J^\alpha_\phi(z) + J^\alpha_h(z) + J^\alpha_{\text{gh}}(z) \qquad \text{for} \ \ \alpha \in H_L \ .
\end{aligned}
\tag{4.4}
$$

The OPEs among the currents are listed in Eq.(3.13). The holomorphic stress energy-momentum tensor of the chiral gauged WZW theory is

$$T_{\rm tot}(z) = T_\phi(z) + T_h(z) + T_{\rm gh}(z) \tag{4.5}$$

where $T_h(z)$ is also given by the Sugawara form

$$\begin{aligned} T_h(z) &= \frac{1}{[-k - 2C_2(H_L)] + C_2(H_L)} \sum_{\alpha=1}^{dim\ H_L} : J_h^\alpha(z) J_h^\alpha(z) : \\ &= \frac{-1}{k + C_2(H_L)} \sum_{\alpha=1}^{dim\ H_L} : J_h^\alpha(z) J_h^\alpha(z) : \end{aligned} \tag{4.6}$$

where the additional minus sign follows from that in Eq.(3.11). The stress energy-momentum tensor of the ghost, $T_{\rm gh}(z)$, is

$$T_{\rm gh}(z) = - \sum_{\alpha=1}^{dim\ H_L} b_z^\alpha(z) \partial_z c^\alpha(z) \ . \tag{4.7}$$

Therefore the central charge of $T_{\rm tot}(z)$ is precisely that for the G/H coset model,

$$\begin{aligned} c_{\rm tot} &= \frac{k \cdot dim(G_L)}{k + C_2(G_L)} + \frac{[-k - 2C_2(H_L)] \cdot dim(H_L)}{-k - C_2(G_L)} - 2 \cdot dim(H_L) \\ &= \frac{k \cdot dim(G_L)}{k + C_2(G_L)} - \frac{k \cdot dim(H_L)}{k + C_2(H_L)} = c_G - c_H \equiv c_{G/H} \end{aligned} \tag{4.8}$$

which was first observed in Ref.[3].

Let us recall from the previous section that the physical primary state, $|\Phi(w, \bar{w}) > \otimes |0>^L_{\rm gh} \otimes |0>^R_{\rm gh}$, in the chiral gauged WZW theory are the BRST singlet states

$$Q^{L,R}_{BRST} \left(|\Phi(w, \bar{w}) > \otimes |0>^L_{\rm gh} \otimes |0>^R_{\rm gh} \right) = 0 \tag{4.9}$$

where $|0>^{L,R}_{\rm gh}$ are the ghost vaccum satisfying, for $\alpha = 1, 2, ..., dimH_L$,

$$c_n^\alpha |0>^L_{\rm gh} = 0 \quad n \geq 1 \ , \quad \text{and} \quad b_n^\alpha |0>^L_{\rm gh} = 0 \quad n \geq 0 \ . \tag{4.10}$$

The state $|0>^L_{\rm gh}$ is also annihilated by the zero mode of the ghost current, $J^\alpha_{\rm gh,0} |0>^L_{\rm gh} = 0$. Similar constraints apply to $|0>^R_{\rm gh}$. Therefore from the construction of BRST operators, Eq.(3.15) and (3.17), we obtain

$$\begin{aligned} J^\alpha_{\rm tot}(z) \Phi(w, \bar{w}) &= \left(J^\alpha_\phi(z) + J^\alpha_h(z) \right) \Phi(w, \bar{w}) = \text{reg. terms} \\ \bar{J}^\beta_{\rm tot}(\bar{z}) \Phi(w, \bar{w}) &= \left(\bar{J}^\beta_\phi(\bar{z}) + \bar{J}^\beta_{\tilde{h}}(\bar{z}) \right) \Phi(w, \bar{w}) = \text{reg. terms} \end{aligned} \tag{4.11}$$

where $\beta = 1, 2, ..., dim\ H_R$. The physical primary fields in the chiral gauged WZW theory can be considered as the WZW primary fields dressed up by a cloud of the gauge fields, $h_{\rm p}$ and $\widetilde{h}_{\rm p}$,

$$\Phi^{\Lambda,\bar{\Lambda}}_{\lambda,\bar{\lambda}}(w,\bar{w}) = F_1(h_{\rm p})(w,\bar{w}) \left(G^{\Lambda}_{\lambda}(w) G^{\bar{\Lambda}}_{\bar{\lambda}}(\bar{w}) \right) F_2(\widetilde{h}_{\rm p})(w,\bar{w}). \tag{4.12}$$

The above factorization property is also justified from the independence of $J^{\alpha}_{\phi}(z)$ and $J^{\alpha}_{h}(z)$ (also $\bar{J}^{\alpha}_{\phi}(\bar{z})$ and $J^{\alpha}_{\tilde{h}}(\bar{z})$). Since the $h_{\rm p}$ field only appears in the holomorphic currents, $J^{\alpha}_{\rm tot}(z)$ and $T_{\rm tot}(z)$, and similarly the $\tilde{h}_{\rm p}$ field only appears in the anti-holomorphic currents, $\bar{J}^{\alpha}_{\rm tot}(\bar{z})$ and $\bar{T}_{\rm tot}(\bar{z})$, it is natural to choose the dressing $F_1(h_{\rm p})$ as a holomorphic function $F_1(h_{\rm p})(z)$ and the dressing $F_2(\tilde{h}_{\rm p})$ as an anti-holomorphic function $F_2(\tilde{h}_{\rm p})(\bar{z})$. In other words, Eq.(4.11) can be further simplified as

$$\begin{aligned} \left(J^{\alpha}_{\phi}(z) + J^{\alpha}_{h}(z)\right) \Phi^{\Lambda}_{\lambda}(w) &= \text{reg. terms} \\ \left(\bar{J}^{\beta}_{\phi}(\bar{z}) + J^{\beta}_{\tilde{h}}(\bar{z})\right) \bar{\Phi}^{\bar{\Lambda}}_{\bar{\lambda}}(\bar{w}) &= \text{reg. terms} \end{aligned} \tag{4.13}$$

where

$$\Phi^{\Lambda}_{\lambda}(w) = H^{\Lambda}_{\lambda}(w) \cdot G^{\Lambda}_{\lambda}(w)\ , \qquad \bar{\Phi}^{\bar{\Lambda}}_{\bar{\lambda}}(\bar{w}) = \bar{H}^{\bar{\Lambda}}_{\bar{\lambda}}(\bar{w}) \cdot \bar{G}^{\bar{\Lambda}}_{\bar{\lambda}}(\bar{w})\ . \tag{4.14}$$

and $H^{\Lambda}_{\lambda}(w)$ satisfy

$$J^{\alpha}_{h}(z) H^{\Lambda}_{\lambda}(w) = \frac{-(t^{\alpha}_{\Lambda,H})_{\lambda,\lambda'} H^{\Lambda}_{\lambda'}(w)}{z-w} + \text{reg. terms} \tag{4.15}$$

where $(t^{\alpha}_{\Lambda,H})_{\lambda,\lambda'} = (t^{\alpha}_{\Lambda})_{\lambda,\lambda'}$ if $\alpha = 1, 2, ...,\ dim\ H_L$. To illustrate Eq.(4.15), consider $G = SU(3)$, $H = SU(2)$ and $\Lambda =$ the fundamental representation of $SU(3)$. The t^{α}_{Λ}'s defined in Eq.(4.1) are the Gell-Mann matrices, where $a = 1, 2, ..., 8$. Then the matrices $t^{\alpha}_{\Lambda,H}$ for $\alpha = 1, 2, 3$, can be decomposed into a block-diagonal form in terms of the representation matrices of $SU(2)$,

$$\left(t^{\alpha}_{\Lambda,H}\right) \equiv \left(t^{\alpha}_{j=1/2}\right) \ \oplus \ \left(t^{\alpha}_{j=0}\right) \tag{4.16}$$

where $\left(t^{\alpha}_{j}\right)$ are the spin-j representation matrices of $SU(2)$ with dimension $(2j+1)\times(2j+1)$. Here $\left(t^{\alpha}_{j=1/2}\right)$ are Pauli matrices and $\left(t^{\alpha}_{j=0}\right) = 0$. $\bar{H}^{\bar{\Lambda}}_{\bar{\lambda}}(\bar{w})$'s satisfy similar constraints. The minus sign in front of $\left(t^{\alpha}_{\Lambda,H}\right)$ indicates that $H^{\Lambda}_{\lambda}(w)$'s are time-like fields. The highest weight of $H^{\Lambda}_{\lambda}(w)$ with respect to the stress energy-momentum, $T_h(z)$, Eq.(4.6), is

$$T_h(z) H^{\Lambda}_{\lambda}(w) = \frac{-\Delta^{\lambda}_{H_L}}{(z-w)^2} H^{\Lambda}_{\lambda}(w) + ... \tag{4.17}$$

where $\Delta_{H_L}^{\lambda} = C_\lambda(H_L)/(k + C_2(H_L))$. $C_\lambda(H_L)$ means symbolically the quadratic Casimir of the representation that λ belongs to. Note that the additional minus sign in Eq.(4.17) follows from Eq.(4.6). Therefore the highest weight of $\Phi_\lambda^\Lambda(w)$ is

$$T_{\rm tot}(z)\Phi_\lambda^\Lambda(w) = \frac{\Delta_G^\Lambda - \Delta_{H_L}^\lambda}{(z-w)^2}\Phi_\lambda^\Lambda(w) + ... \tag{4.18}$$

which is the same as that of the primary fields in the coset theory G/H. In the above example $G/H = SU(3)/SU(2)$, $\Delta_{H_L}^\lambda = \frac{j(j+1)}{k+2}$. Therefore, with respect to T_h, the highest weights of $H_\lambda^\Lambda(w)$, for $\lambda = 1, 2$ and that of $H_{\lambda=3}^\Lambda(w)$ are respectively

$$\begin{aligned} \Delta_{H_L}^{\lambda=1,2} &= \frac{3}{4(k+2)} \\ \Delta_{H_L}^{\lambda=3} &= 0 \ . \end{aligned} \tag{4.19}$$

The holomorphic part of the correlation functions of these dressed primary fields can be calculated as

$$\begin{aligned} &< \Phi_{\lambda_1}^{\Lambda_1}(z_1)\Phi_{\lambda_2}^{\Lambda_2}(z_2)...\Phi_{\lambda_n}^{\Lambda_n}(z_n) > \\ &=< G_{\lambda_1}^{\Lambda_1}(z_1)G_{\lambda_2}^{\Lambda_2}(z_2)...G_{\lambda_n}^{\Lambda_n}(z_n) >< H_{\lambda_1}^{\Lambda_1}(z_1)H_{\lambda_2}^{\Lambda_2}(z_2)...H_{\lambda_n}^{\Lambda_n}(z_n) > \ . \end{aligned} \tag{4.20}$$

Since this expression is in general a linear combination of conformal blocks, we may for convenience consider one conformal block at a time. The primary field G_λ^Λ can be decomposed into $G_\lambda^\Lambda(z) = g_\lambda^\Lambda(z)\Omega_\lambda^\Lambda(z)$ where $\Omega_\lambda^\Lambda(z)$'s satisfy

$$\begin{aligned} J_\phi^\alpha(z)\Omega_\lambda^\Lambda(w) &= \frac{(t_{\Lambda,H}^\alpha)_{\lambda,\lambda'}\Omega_{\lambda'}^\Lambda(w)}{z-w} + \text{reg. terms} \\ J_\phi^\alpha(z)g_\lambda^\Lambda(w) &= \text{reg. terms} \qquad\qquad \text{for } \alpha = 1, 2, ..., dim\ H_L \end{aligned} \tag{4.21}$$

where $(t_{\Lambda,H}^\alpha)$ is the same as that in Eq.(4.15).

Because of the time-like nature of the dressed field H_λ^Λ (due to the additional minus sign in front of $I(h_{\rm p})$ in Eq.(3.11)), the second factor on the right-hand side of Eq.(4.20) is the reciprocal of the conformal block of the corresponding fields in the representation of the subgroup H_L, *i.e.*

$$< H_{\lambda_1}^{\Lambda_1}(z_1)H_{\lambda_2}^{\Lambda_2}(z_2)...H_{\lambda_n}^{\Lambda_n}(z_n) >= \Big\{< \Omega_{\lambda_1}^{\Lambda_1}(z_1)\Omega_{\lambda_2}^{\Lambda_2}(z_2)...\Omega_{\lambda_n}^{\Lambda_n}(z_n) >\Big\}^{-1} \ . \tag{4.22}$$

Therefore Eq.(4.20) is equal to the conformal block in the coset theory G/H, *i.e.*

$$< \Phi_{\lambda_1}^{\Lambda_1}(z_1)\Phi_{\lambda_2}^{\Lambda_2}(z_2)...\Phi_{\lambda_n}^{\Lambda_n}(z_n) >=< g_{\lambda_1}^{\Lambda_1}(z_1)g_{\lambda_2}^{\Lambda_2}(z_2)...g_{\lambda_n}^{\Lambda_n}(z_n) > \tag{4.23}$$

The antiholomorphic parts can be similarly calculated. To obtain the correlation function, we have to impose the monodromy invariant and fusion invariant conditions in combining the holomorphic and the anti-holomorphic conformal blocks. For simple theories, consistent correlation functions exist only for $H_L = H_R$, which is equivalent to the vector gauged WZW theories. However, in the construction of heterotic type of string models[23], there exists consistent left-right asymmetric correlation functions which can be possibly provided by the chiral gauged WZW theories.

To illustrate in more detail the general formalism presented above, two examples are given in the following, *i.e.* Z_k parafermion (PF) (*i.e.* $SU(2)_k/U(1)$ coset theory) and $SU(2)_k \otimes SU(2)_l/SU(2)_{k+l}$ coset model.

(I.) $\mathbf{G} = SU(2)_k, \;\; \mathbf{H} = U(1):$

For the sake of concreteness, we briefly summarize some results in the $SU(2)_k$ WZW theory[7]. The holomorphic $SU(2)_k$ KM current algebra has the following OPEs among its dimension-one currents:

$$\begin{aligned} J^+_\phi(z)J^-_\phi(w) &= \frac{k}{(z-w)^2} + \frac{2J^3_\phi(w)}{z-w} + \text{reg. terms} \\ J^3_\phi(z)J^\pm_\phi(w) &= \frac{\pm J^\pm_\phi(w)}{z-w} + \text{reg. terms} \\ J^3_\phi(z)J^3_\phi(w) &= \frac{k/2}{(z-w)^2} + \text{reg. terms} \end{aligned} \tag{4.24}$$

The holomorphic stress energy-momentum tensor is given by the Sugawara form

$$T_{SU(2)}(z) = \frac{1}{k+2}\sum_a : J^a_\phi(z)J^a_\phi(z) : \tag{4.25}$$

whose central charge is $c_{SU(2)} = 3k/(k+2)$. The antiholomorphic currents, $\bar{J}^3_\phi(\bar{z})$ and $\bar{J}^\pm_\phi(\bar{z})$, commute with the holomorphic currents and obey a similar algebra. The primary fields in the $SU(2)_k$ WZW theory are denoted by $G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z})$, satisfying

$$\begin{aligned} &J^a_{\phi,n}G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z}) = 0 \;\; = \bar{J}^a_{\phi,n}G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z}), \qquad\quad \text{for } n>0 \\ &J^+_{\phi,0}G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z}) = 0 \;\; = \bar{J}^+_{\phi,0}G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z}) \\ &J^3_{\phi,0}G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z}) = jG^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z})\;, \qquad \bar{J}^3_{\phi,0}G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z}) = \bar{j}G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z}) \end{aligned} \tag{4.26}$$

where $J^a_{\phi,n}$ and $\bar{J}^a_{\phi,n}$ are the moding operators defined by $J^a_\phi(z) = \sum_n J^a_{\phi,n}z^{-n-1}$ and $\bar{J}^a_\phi(\bar{z}) = \sum_n \bar{J}^a_{\phi,n}\bar{z}^{-n-1}$. The Virasoro primary fields, $G^{j,\bar{j}}_{m,\bar{m}}(z,\bar{z})$, are defined by

$$G^{j,\bar{j}}_{m,\bar{m}}(z,\bar{z}) \;=\; (J^-_{\phi,0})^{j-m}(\bar{J}^-_{\phi,0})^{\bar{j}-\bar{m}}G^{j,\bar{j}}_{j,\bar{j}}(z,\bar{z}) \tag{4.27}$$

whose $J^3_{\phi,0}$ quantum number is m and the highest weight is $\Delta^j = j(j+1)/(k+2)$. As discussed above, the primary fields can be separated into the holomorphic and antiholomorphic parts, *i.e.* $G^{j,\bar{j}}_{m,\bar{m}}(z,\bar{z}) = G^j_m(z)\bar{G}^{\bar{j}}_{\bar{m}}(\bar{z})$.

Now let us consider the chiral gauged WZW theory where $H_L = U(1)$. The left $U(1)$ gauge field $A^L_{\bar{z}}(\equiv h^{-1}\partial_{\bar{z}}h)$ has the action as shown in Eq.(3.11)

$$S(h) = -kI(h) = \frac{-k}{16\pi}\int d^2z\partial_\mu h^{-1}\partial^\mu h = \frac{-1}{8\pi}\int d^2z\partial_z\sigma\partial_{\bar{z}}\sigma \tag{4.28}$$

where the WZ term vanishes and $C_2(U(1)) = 0$. In Eq.(4.28), h has been parametrized by

$$h = \exp\left(-i\sigma(z,\bar{z})/\sqrt{k}\right). \tag{4.29}$$

Eq.(4.28) is the action of a free time-like boson with the equation of motion, $\partial_z\partial_{\bar{z}}\sigma(z,\bar{z}) = 0$. Therefore we can separate the holomorphic and antiholomorphic parts, $\sigma(z,\bar{z}) \equiv \sigma(z) + \bar{\sigma}(\bar{z})$. The 2-point functions of $\sigma(z)$ and $\bar{\sigma}(\bar{z})$ can be normalized as

$$< \sigma(z)\sigma(w) >= 2\log(z-w), \quad < \bar{\sigma}(\bar{z})\bar{\sigma}(\bar{w}) >= 2\log(\bar{z}-\bar{w}). \tag{4.30}$$

Hence h becomes

$$h = \exp\left(-i\sigma(z)/\sqrt{k}\right)\exp\left(-i\bar{\sigma}(\bar{z})/\sqrt{k}\ .\right) \tag{4.31}$$

A chiral gauge transformation, $g_L^{-1}(\bar{z}) = \exp\left(-i\bar{\sigma}(\bar{z})/\sqrt{k}\right)$, can be used to remove the gauge degree of freedom such that $h(z,\bar{z})$ becomes a holomorphic function, $h_{\rm p}(z)$,

$$h_{\rm p}(z) = \exp\left(-i\sigma(z)/\sqrt{k}\right). \tag{4.32}$$

The $U(1)$ current J_h from the action Eq.(4.28) is defined by (using the formula shown in the paragraph below Eq.(2.16))

$$J_h(z) = \frac{-1}{2}\cdot(-k)\cdot\partial_z h\cdot h^{-1} = \frac{-i}{2}\sqrt{k}\partial_z\sigma(z) \tag{4.33}$$

which satisfies $U(1)$ KM algebra at the level "$-k$", justifying the action Eq.(4.28) we started with,

$$J_h(z)J_h(w) = \frac{-k/2}{(z-w)^2} + \text{reg. terms}. \tag{4.34}$$

J_h commutes with the original $SU(2)_k$ KM algebra, *i.e.*

$$J_h(z)J^{3,\pm}_\phi(w) = \text{reg. terms}. \tag{4.35}$$

The σ field has the stress energy-momentum tensor $T_\sigma(z) = \frac{1}{4}\partial_z\sigma\partial_z\sigma$ with central charge $c_\sigma = 1$. The b, c ghost action in Eq.(3.11) has the stress energy-momentum tensor $T_{bc}(z) = -b\partial_z c$ whose central charge is $c_{\rm gh} = -2$. Here $J_{\rm gh}$ vanishes because of the abelian nature of H_L. Therefore,

$$\begin{aligned} & J^3_{\rm tot}(z) = J^3_\phi(z) + J_h(z) \\ \text{where} \quad & J^3_{\rm tot}(z) J^3_{\rm tot}(w) \sim \text{reg. terms.} \end{aligned} \tag{4.36}$$

Note that the KM anomaly term is absent as is required by the gauge invariance. The stress energy-momentum tensor of the left-moving gauged WZW theory is obtained as

$$T_{\rm tot}(z) = T_{SU(2)}(z) + T_\sigma(z) + T_{bc}(z) \tag{4.37}$$

whose central charge is $c_{\rm tot} = 3k/(k+2) + 1 - 2 = 2(k-1)/(k+2)$. The above discussion applies equally to the right-handed sector if $H_R = U(1)$ also, where

$$\widetilde{h}_{\rm p}(\bar{z}) = \exp\left(-i\widetilde{\sigma}(\bar{z})/\sqrt{k}\right) \quad \text{and} \quad J_{\tilde{h}}(\bar{z}) = \frac{-i}{2}\sqrt{k}\partial_{\bar{z}}\widetilde{\sigma}(\bar{z}) \ . \tag{4.38}$$

Following from the discussion between Eq.(4.11) and (4.13), we have to solve

$$\begin{aligned} \left(J^3_\phi(z) + J_h(z)\right) \Phi^j_m(w) &= \text{reg. terms} \\ \left(\bar{J}^3_\phi(\bar{z}) + J_{\tilde{h}}(\bar{z})\right) \bar{\Phi}^{\bar{j}}_{\bar{m}}(\bar{w}) &= \text{reg. terms} \end{aligned} \tag{4.39}$$

where

$$\Phi^j_m(z) = F_1(h_{\rm p})(z) \cdot G^j_m(z) \ , \qquad \bar{\Phi}^{\bar{j}}_{\bar{m}}(\bar{z}) = F_2(\widetilde{h}_{\rm p})(\bar{z}) \bar{G}^{\bar{j}}_{\bar{m}}(\bar{z}) \ . \tag{4.40}$$

The answers are

$$\begin{aligned} \Phi^j_m(z) &= G^j_m(z) \exp\left(-im\sigma(z)/\sqrt{k}\right) = G^j_m(z) h^m_{\rm p}(z) \\ \bar{\Phi}^{\bar{j}}_{\bar{m}}(z) &= \bar{G}^{\bar{j}}_{\bar{m}}(\bar{z}) \exp\left(-im\widetilde{\sigma}(\bar{z})/\sqrt{k}\right) = \bar{G}^{\bar{j}}_{\bar{m}}(\bar{z}) \widetilde{h}^m_{\rm p}(\bar{z}) \ . \end{aligned} \tag{4.41}$$

The holomorphic part of the correlation functions of these dressed primary fields can be calculated as

$$\begin{aligned} &< \Phi^{j_1}_{m_1}(z_1)\Phi^{j_2}_{m_2}(z_2)...\Phi^{j_n}_{m_n}(z_n) > = < G^{j_1}_{m_1}(z_1) G^{j_2}_{m_2}(z_2)...G^{j_n}_{m_n}(z_n) > \cdot \\ &\quad < e^{\left(-im_1\sigma(z_1)/\sqrt{k}\right)} e^{\left(-im_2\sigma(z_2)/\sqrt{k}\right)} ... e^{\left(-im_n\sigma(z_n)/\sqrt{k}\right)} > \end{aligned} \tag{4.42}$$

where the first factor is the correlation function of $SU(2)_k$ WZW theory. These correlation functions are exactly the same as those in the Z_k parafermion (PF) theory[10]. To show this more explicitly, we can use the Z_k PF as a particular parametrization of the $SU(2)_k$ KM currents (the antiholomorphic part is again omitted),

$$
\begin{aligned}
J^+(z) &= \sqrt{k}\psi_1(z)\exp\left(i\rho(z)/\sqrt{k}\right)\\
J^-(z) &= \sqrt{k}\psi_1^\dagger(z)\exp\left(-i\rho(z)/\sqrt{k}\right)\\
J^3(z) &= \frac{i}{2}\sqrt{k}\partial_z\rho(z)
\end{aligned}
\tag{4.43}
$$

where ψ_1 and $\psi_1^\dagger$ are the PF currents which are the symmetry currents in the Z_k PF theory. $\rho(z)$ is a free space-like boson with the two point function

$$
< \rho(z)\rho(w) >= -2\log(z-w)\ . \tag{4.44}
$$

The central charge of this Z_k PF is $c_{\rm PF} = 2(k-1)/(k+2)$ which is the same as that for Eq.(4.37). The primary fields Ψ^j_m in the Z_k PF theory are also related to those in the WZW theory as

$$
G^j_m(z) = \Psi^j_m(z)\exp\left(im\rho(z)/\sqrt{k}\right) \tag{4.45}
$$

whose highest weight is

$$
\Delta\left(\Psi^j_m\right) = \Delta^j - m^2/k\ . \tag{4.46}
$$

Hence it can be shown that

$$
\begin{aligned}
&< \Phi^{j_1}_{m_1}(z_1)\Phi^{j_2}_{m_2}(z_2)...\Phi^{j_n}_{m_n}(z_n) >\\
&=< \Psi^{j_1}_{m_1}(z_1)\Psi^{j_2}_{m_2}(z_2)...\Psi^{j_n}_{m_n}(z_n) >< e^{\left(im_1\rho(z_1)/\sqrt{k}\right)}e^{\left(im_2\rho(z_2)/\sqrt{k}\right)}...e^{\left(im_n\rho(z_n)/\sqrt{k}\right)} >\\
&\quad < e^{\left(-im_1\sigma(z_1)/\sqrt{k}\right)}e^{\left(-im_2\sigma(z_2)/\sqrt{k}\right)}...e^{\left(-im_n\sigma(z_n)/\sqrt{k}\right)} >
\end{aligned}
\tag{4.47}
$$

where

$$
< e^{\left(-im_1\sigma(z_1)/\sqrt{k}\right)}e^{\left(-im_2\sigma(z_2)/\sqrt{k}\right)}...e^{\left(-im_n\sigma(z_n)/\sqrt{k}\right)} >= \prod_{i=1}^{n}\prod_{j>i}(z_i-z_j)^{-2m_im_j/k} \tag{4.48}
$$

and

$$
< e^{\left(im_1\rho(z_1)/\sqrt{k}\right)}e^{\left(im_2\rho(z_2)/\sqrt{k}\right)}...e^{\left(im_n\rho(z_n)/\sqrt{k}\right)} >= \prod_{i=1}^{n}\prod_{j>i}(z_i-z_j)^{2m_im_j/k} \tag{4.49}
$$

according to the 2-point functions in Eq.(4.30) and (4.44). This demonstrates that

$$
< \Phi^{j_1}_{m_1}(z_1)\Phi^{j_2}_{m_2}(z_2)...\Phi^{j_n}_{m_n}(z_n) >=< \Psi^{j_1}_{m_1}(z_1)\Psi^{j_2}_{m_2}(z_2)...\Psi^{j_n}_{m_n}(z_n) >\ . \tag{4.50}
$$

Note that the highest weight of $\Phi^j_m(z)$ can be calculated as

$$\left(T_{SU(2)}(z) + T_\sigma(z) + T_{\rm gh}(z)\right)\Phi^j_m(w) = \frac{(\Delta_j - m^2/k)}{(z-w)^2}\Phi^j_m(w) + ... \tag{4.51}$$

which is the same as that in Eq.(4.46). This establishes the isomorphism between the chiral gauged WZW theory and the Z_k PF theory in the CFT. In this picture of the gauged WZW theory, the gauge fields form a "cloud" surrounding the primary fields in the WZW theory and screening the $U(1)$ subgroup of the $SU(2)$ WZW theory. According to Eq.(4.30), $h_{\rm p}$ is treated as a non-unitary theory based on a time-like boson, in contrast to the space-like boson defined in Eq.(4.44). In other words, the gauge fields cancel one physical unitary degree of freedom from the original WZW theory, leaving the unitary Z_k PF behind. The original $SU(2)$ symmetry is no longer present in the gauged WZW theory, which can be understood when applying $J^\pm_0$ to Φ^j_m. The resulting states, $J^\pm_0\Phi^j_m(z) = G^j_{m\pm1}h^m_{\rm p}$, do not satisfy the physical state condition, Eq.(4.39). This implies that the "physical currents", $J^\pm_{\rm p}$, in the chiral gauged WZW theory are also screened by a cloud of gauge fields. The simple requirement is

$$J^\pm_{\rm p}(z)\Phi^j_m(w) \sim \Phi^j_{m\pm1}(w)\ . \tag{4.52}$$

This implies

$$J^\pm_{\rm p}(z) = J^\pm(z)\exp\left(\mp i\sigma(z)/\sqrt{k}\right). \tag{4.53}$$

It can be checked that

$$J^3_{\rm tot}(z)J^\pm_{\rm p}(w) = \text{reg. terms} \tag{4.54}$$

which means that the negative modings of $J^\pm_{\rm p}$ can be used to act on the physical primary fields and generate the descendants in the gauged WZW theory. The OPE between $J^+_{\rm p}(z)$ and $J^-_{\rm p}(w)$ can be calculated as

$$\begin{aligned}
J^+_{\rm p}(z)J^-_{\rm p}(w) &= \left(J^+(z)J^-(w)\right)\left(\exp\left(-i\sigma(z)/\sqrt{k}\right)\exp\left(i\sigma(w)/\sqrt{k}\right)\right) \\
&= \left(\frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w} + ...\right)(z-w)^{2/k}\left[1 - \frac{i}{\sqrt{k}}(z-w)\partial\sigma(w)\right] \\
&= (z-w)^{-2+2/k}\left[1 + ...\right]
\end{aligned} \tag{4.55}$$

which is isomorphic to that of the Z_k PF currents ψ_1 and $\psi^\dagger_1$. This concludes our argument that the chiral $U(1)$ gauged $SU(2)_k$ WZW theory is isomorphic to the Z_k PF theory. The above argument can easily generalize to the case where both H_L and H_R are abelian.

(II.) $\mathbf{G} = SU(2)_k \otimes SU(2)_l$, $\mathbf{H} = SU(2)$**:**

In the case where G is not simple but semi-simple, the above analysis must be slightly modified. To be specific, let us consider the case when $G = SU(2)_k \otimes SU(2)_l$ where both k and l are integers. In this case, the WZW action can be written as

$$S(\phi_1, \phi_2) = kI(\phi_1) + lI(\phi_2). \tag{4.56}$$

The chiral gauge coupling, $H = SU(2)$, can be introduced as that in Eq.(2.35)

$$S(A^L_{\bar{z}}, A^R_z, \phi_1, \phi_2) = S(A^L_{\bar{z}}, A^R_z, \phi_1) + S(A^L_{\bar{z}}, A^R_z, \phi_2)\ . \tag{4.57}$$

Then Eq.(3.4) for this case becomes

$$S(h, \widetilde{h}, \phi_1, \phi_2) = kI(\widehat{\phi}_1) + lI(\widehat{\phi}_2) - (k+l)I(h) - (k+l)I(\widetilde{h}). \tag{4.58}$$

The Kač$-$Moody currents from $\phi_1, \phi_2, h_{\rm p}$ and $\widetilde{h}_{\rm p}$ and the associated Sugawara stress energy-momentum tensors can be similarly constructed as discussed before. Therefore, we can first calculate the total central charge for the holomorphic sector,

$$\begin{aligned} c_{\rm tot} &= \frac{3k}{k+2} + \frac{3l}{l+2} + \frac{3(-k-l-4)}{-k-l-2} - 2\times 3 \\ &= \frac{3k}{k+2} + \frac{3l}{l+2} - \frac{3(k+l)}{k+l+2} \end{aligned} \tag{4.59}$$

which is the same as that of the $SU(2)_k \otimes SU(2)_l / SU(2)_{k+l}$ theory. Before the introduction of gauge couplings, the primary field of $SU(2)_k \times SU(2)_l$ is

$$G^{j_1 j_2 \bar{j}_1 \bar{j}_2}(z, \bar{z}) = \left(G^{j_1}(z) G^{j_2}(z)\right) \left(\bar{G}^{\bar{j}_1}(\bar{z}) G^{\bar{j}_2}(\bar{z})\right) \tag{4.60}$$

where the m quantum number is suppressed, *i.e.* $G^j(z) = G^j_j(z)$.

Let us consider the sector $\left[G^{j_1}(z) G^{j_2}(z)\right]$ which includes both the primary field and its current algebra descendants. From Eq.(4.58), we see that the gauge field's contribution behaves like a $SU(2)_{k+l}$ WZW theory with time-like metric. Since the $\left[G^{j_1}(z) G^{j_2}(z)\right]$ sector is reducible in terms of the representations of $SU(2)_{k+l}$ WZW theory, this means the $\left[G^{j_1}(z) G^{j_2}(z)\right]$ sector contains subsectors, each of which will have different gauge dressing when we turn on the gauge couplings. Therefore, we have to decompose the sector $\left[G^{j_1}(z) G^{j_2}(z)\right]$ into subsectors

$$\left[G^{j_1}(z) G^{j_2}(z)\right] = \sum_j V^j_{j_1 j_2}(z) \tag{4.61}$$

where generically $j = 0, 1/2, ..., (k+l)/2$, with the restriction $j = j_1 + j_2 \ (mod\, 1)$. Therefore the conformal dimension of $V^j_{j_1 j_2}(z)$ is

$$\Delta\left(V^j_{j_1 j_2}\right) = \frac{j_1(j_1+1)}{k+2} + \frac{j_2(j_2+1)}{l+2} + N^j_{j_1 j_2} \tag{4.62}$$

where $N^j_{j_1 j_2}$ is a non-negative integer.

Following the discussion between Eq.(4.11) and (4.15), we find the dressed primary fields to be

$$\Phi^j_{j_1 j_2}(z) = V^j_{j_1 j_2}(z) H^j_{j_1+j_2}(z) \tag{4.63}$$

where the dressing field $H^j_{j_1+j_2}(z)$ is a time-like primary field in the spin j representation of $SU(2)_{k+l}$ theory, and $H^j_{j_1+j_2}(z)$ has the conformal dimension $-j(j+1)/(k+l+2)$. Hence the conformal dimension of $\Phi^j_{j_1 j_2}(z)$ is given by

$$\Delta(\Phi^j_{j_1 j_2}) = \frac{j_1(j_1+1)}{k+2} + \frac{j_2(j_2+1)}{l+2} - \frac{j(j+1)}{k+l+2} + N^j_{j_1 j_2} \tag{4.64}$$

which is the same as that of the primary field in the $SU(2)$ coset theory. The holomorphic correlation function of these dressed primary fields are given as

$$\begin{aligned} &< \Phi^{j(1)}_{j_1(1) j_2(1)}(z_1) \Phi^{j(2)}_{j_1(2) j_2(2)}(z_2) ... \Phi^{j(n)}_{j_1(n) j_2(n)}(z_n) >= \\ &\quad < V^{j(1)}_{j_1(1) j_2(1)}(z_1) V^{j(2)}_{j_1(2) j_2(2)}(z_2) ... V^{j(n)}_{j_1(n) j_2(n)}(z_n) > \cdot \\ &\quad < H^{j(1)}_{j_1(1)+j_2(1)}(z_1) H^{j(2)}_{j_1(2)+j_2(2)}(z_2) ... H^{j(n)}_{j_1(n)+j_2(n)}(z_n) > \end{aligned} \tag{4.65}$$

which is the same as that of the $SU(2)_k \otimes SU(2)_l / SU(2)_{k+l}$ coset theory. To see that $< \Phi^{j(1)}_{j_1(1) j_2(1)}(z_1) \Phi^{j(2)}_{j_1(2) j_2(2)}(z_2) ... \Phi^{j(n)}_{j_1(n) j_2(n)}(z_n) >$ is the correlation function for the coset theory, we can decompose the WZW primary field,

$$V^j_{j_1 j_2}(z) = g^j_{j_1 j_2}(z) G^j_{j_1+j_2}(z) \tag{4.66}$$

where $g^j_{j_1 j_2}(z)$ is the coset primary field and $G^j_{j_1+j_2}(z)$ is the primary field with m quantum number $j_1 + j_2$ in the spin-j representation of $SU(2)_{k+l}$ WZW theory, whose conformal dimension is $j(j+1)/(k+l+2)$, which is exactly opposite to that of $H^j_{j_1+j_2}(z)$. From the argument leading to Eq.(4.23), we can see that the holomorphic correlation function, Eq.(4.65), is precisely that of the $g^{j(i)}_{j_1(i) j_2(i)}(z_i)$, *i.e.* the holomorphic correlation function of primary fields of the $SU(2)_k \otimes SU(2)_l / SU(2)_{k+l}$ coset theory.

5. Explicit Representations of the Matter Fields

In this section, two explicit examples are briefly discussed to illustrate some of the issues in the chiral gauged theory. The first one is the $SO(N)$ representation of real fermions in which we can see why chiral gauge coupling is anomaly-free. The other one is a free spin $(1,0)$ system of ${\omega^\dagger}^a$, ω^a in the adjoint representation of G_L and couples to chiral gauge fields in $H_L = G_L$; this is an example of the critical k case in which the $h_{\rm p}$ fields decouple.

As an example of Eq.(2.1), real fermions with the chiral gauge couplings in two dimensions can be introduced with the action

$$S = \frac{1}{8\pi}\int \psi^i(\delta^{ij}\partial_{\bar z} + {A^L_{\bar z}}^a t^a_{ij})\psi^j \tag{5.1}$$

where $i = 1, 2, ..., N$ for $N \geq 4$. We can consider these fermions forming a vector representation of $SO(N)$ with the representation matrix t^a. $a = 1, 2, ..., dim\,(SO(N))$ for each generator. The affine KM currents can be constructed as

$$J^a(z) \equiv \psi^i(z)t^a_{ij}\psi^j(z) \tag{5.2}$$

satisfying $SO(N)$ affine KM algebra at the level $k = 1$. Here the gauge couplings are chiral, *i.e.* in Eq.(5.1) $A^L = \left(0, A^L_{\bar z}\right)$ and A^R is absent.

Before we consider the vacuum polarization of this theory, let us first consider the general current-current correlation function, $< J^a_\mu J^b_\nu >= \Pi^{ab}_{\mu\nu} = \delta^{ab}\Pi_{\mu\nu}$, which has the general form,

$$\Pi_{\mu\nu}(q) = g_{\mu\nu}\Pi_1(q) + \frac{q_\mu q_\nu}{q^2}\Pi_2(q) + \frac{\epsilon_{\mu\lambda}q^\lambda q_\nu + \epsilon_{\nu\lambda}q^\lambda q_\mu}{2q^2}\Pi_3(q) \tag{5.3}$$

where q_μ is the momentum and $\Pi_i(q)$ are functions of q^2. As is clear from the form of Eq.(5.3), it is impossible for all (vector, axial, chiral) currents to be conserved. For example, the requirement of the vector current conservation demands

$$q^\mu\Pi_{\mu\nu}(q) = 0 \Rightarrow \Pi_1(q) = -\Pi_2(q) \quad \text{and} \quad \Pi_3(q) = 0 \tag{5.4}$$

while the conservation of the axial-vector current implies $q_\lambda\epsilon^{\lambda\mu}\Pi_{\mu\nu}(q) = 0$ which demands

$$\Pi_1 = 0 \quad \text{and} \quad \Pi_3 = 0\ . \tag{5.5}$$

Hence the demand of both conservation implies $\Pi_1 = \Pi_2 = \Pi_3 = 0$. An examination of the one-loop diagram shows that we can regularize the one-loop effect such that either (5.4) or (5.5) can be satisfied. This is sometimes called the renormalization condition. But there does not exist a regularization choice such that all $\Pi_i(q)$ vanishes. Therefore if the fermions are coupled only to vector or only to axial vector currents, the gauge invariance can be maintained, but not when coupled to both.

For the theory Eq.(5.1), only J^a_z couples to $A^L_{\bar{z}}$, so Π^{ab}_{zz} is the vacuum polarization from the fermion loop contributions. The chiral gauge invariance requires that

$$0 = q_{\bar{z}}\Pi_{zz}(q) = \frac{q_z}{2}\left[\Pi_2(q) - \Pi_3(q)\right] \tag{5.6}$$

where the coordinates $g^{z\bar{z}} = 1$ and $\epsilon^{z\bar{z}} = 1$ are used. Therefore, the renormalization condition can be chosen as $\Pi_2(q) = \Pi_3(q)$ such that the chiral gauge invariance is preserved. Since in this case $C_2(SO(N)) = N - 2$, the full quantum action has to include h_{p} field as discussed in Sec.3.

The next example is the free ghost system of spins $(1,0)$, ${\omega^\dagger}^a$, ω^a. Their action is

$$S_L = \int d^2z \prod_{a=1}^{dimG_L} {\omega^\dagger}^a\ \bar{\partial}\ \omega^a\ . \tag{5.7}$$

The equations of motion are $\bar{\partial}\omega^a = 0 = \bar{\partial}{\omega^\dagger}^a$. The non-trivial OPEs between them are

$$\omega^a(z)\ {\omega^\dagger}^b(w) = \frac{\delta^{ab}}{z-w} = -\ {\omega^\dagger}^a(z)\ \omega^b(w) \tag{5.8}$$

and the rest are regular. The action, (5.7), has a local symmetry determined by

$$\begin{aligned} &g_L(z) \equiv \exp\left(\ v_L(z)\right) \quad \text{with} \quad v_L(z)_{bc} = \epsilon^a(z) f^a_{bc} \\ &\text{and} \qquad\quad \omega^\dagger \rightarrow \omega^\dagger g_L^{-1}(z)\ , \quad\ \omega \rightarrow g_L(z)\omega\ . \end{aligned} \tag{5.9}$$

The infinitesimal variations of the fields are

$$\begin{aligned} \delta^a\omega^b(z) &= \epsilon^a(z) f^a_{bc}\omega^c(z) \equiv \epsilon^a(z)\left[Q^a\ ,\ \ \omega^b(z)\right] \\ \delta^a{\omega^\dagger}^b(z) &= \epsilon^a(z) f^a_{bc}{\omega^\dagger}^c(z) \equiv \epsilon^a(z)\left[Q^a\ ,\ \ {\omega^\dagger}^b(z)\right] \end{aligned} \tag{5.10}$$

where Q^a is the generator of the global transformation in Eq.(5.9) and the index a is not summed over. From the OPEs in Eq.(5.8), we can easily solve Q^a as the contour integral of the affine KM symmetry current

$$Q^a \equiv \oint dz J^a = \oint dz f^{abc}\omega^b{\omega^\dagger}^c \tag{5.11}$$

These currents satisfy

$$J^a(z)J^b(w) = \frac{-C_2(G_L)}{(z-w)^2} + \frac{f^{abc}J^c(w)}{z-w} + ... \tag{5.12}$$

which is the affine KM algebra at level $k = -2C_2(G_L)$. We can now introduce the chiral gauge symmetry, *i.e.* replacing $\epsilon^a(z)$ by $\epsilon^a(\bar{z})$. The covariant derivative can be introduced as $D_{\bar{z}} \equiv \partial_{\bar{z}} + A^L_{\bar{z}}$. The action becomes

$$S_L = \int d^2z \prod_{a=1}^{dimG_L} \omega^{\dagger a} \ \left(\bar{\partial} + A^L_{\bar{z}}\right) \ \omega^a \ . \tag{5.13}$$

The invariance of the action requires the gauge transformation on $A^L_{\bar{z}}$ as

$$A^L_{\bar{z}} = g_L \ A^L_{\bar{z}} \ g_L^{-1} - \partial_{\bar{z}} g_L \ g_L^{-1} \ . \tag{5.14}$$

Since $k = -2C_2(G_L)$, which is the critical value that $h_{\rm p}$ decouples from the matter system. Therefore the gauge fixed partition function with Fadeev-Popov ghosts $(b^a, \ c^a)$ is

$$\begin{aligned} \mathcal{Z}_{\rm gf} &= \int \mathcal{D}\omega^{\dagger a}\mathcal{D}\omega^a\mathcal{D}b^a\mathcal{D}c^a \exp\left(-S_{gf}\right) \\ \text{where} \quad S_{\rm gf} &= \int d^2z \left(\omega^{\dagger a}\bar{\partial}\omega^a + b^a\bar{\partial}c^a\right) \ . \end{aligned} \tag{5.15}$$

The full quantum action has the BRST symmetry which can be obtained from Eq.(5.9) with the well-known modification, $\epsilon^a \to \lambda c^a$, ($\lambda$ is a constant grassman number.)

$$\begin{aligned} \delta \ \omega^b &= \lambda c^a \delta^a \omega^b = \lambda c^a f^{abc}\omega^c \equiv [Q_{BRST} \ , \ \omega^b] \\ \delta \ \omega^{\dagger b} &= \lambda c^a \delta^a \omega^{\dagger b} = \lambda c^a f^{abc}\omega^{\dagger c} \equiv [Q_{BRST} \ , \ \omega^{\dagger b}] \\ \delta \ c^\beta &= \frac{1}{2}\lambda c^a f^{a\beta c}\gamma^c \equiv \{Q_{BRST} \ , \ c^\beta\} \\ \delta \ b^\beta &= \lambda f^{\beta\alpha\gamma}\omega^\alpha\omega^{\dagger\gamma} + \lambda c^\alpha f^{\alpha\beta\gamma}b^\gamma \equiv \{Q_{BRST} \ , \ b^\beta\} \ . \end{aligned} \tag{5.16}$$

where $\omega, \omega^\dagger, b, c$'s dependence in z is suppressed. It is an easy check that $\delta S_{\rm gf} = 0$. The BRST current and operator can be found if we demand the above variations as the results of the (anti-)commutation relations between the BRST oerators and various fields. The simple calculation from the OPE in (5.8) gives

$$\begin{aligned} Q_{BRST} &= \oint J_{BRST}(z) \\ J_{BRST}(z) &= \sum_{\alpha\beta\gamma} c^\alpha f^{\alpha\beta\gamma}\omega^\beta\omega^{\dagger\gamma}(z) - \frac{1}{2}f^{\alpha\beta\gamma}c^\alpha c^\beta b^\gamma(z) \end{aligned} \tag{5.17}$$

This is exactly the same BRST operator we found from the purely abstract gauged WZW theory, (3.20). Note that this model has $G_L = H_L$ while G_R and H_R are absent.

It is natural to ask whether this model is unitary or not. The $\omega - \omega^\dagger$ theory without gauge fields is of course non-unitary. However, of all the states in the gauged $\omega - \omega^\dagger$ theory, Eq.(5.15), only a small subset of them satisfy the physical state conditions (or equivalently, BRST cohomology). This situation is very similar to the string theory case. Recall that the conformal theory of 26 bosons with Minkowski metric is non-unitary. However, in the bosonic string theory, the spectrum must satisfy the physical state conditions. As a result, the string theory, which is the gauged (with the metric) version of the 26-boson conformal theory, is unitary. Following from this analogy, it is likely that the gauged $\omega - \omega^\dagger$ theory is also unitary. A preliminary analysis seems to support this conjecture[24].

6. Conclusion

In this paper, we showed that the chiral gauged WZW theory is anomaly free and its quantization can be carried out. In the special case where the left and the right chiral gauge groups are the same, the theory is in agreement with the vector gauged WZW theory. However, the chiral gauged WZW theory has the advantage of allowing the existence of the left-right asymmetrical gauge groups which may be suitable for the heterotic type model building in the fractional superstring theory. By an examination of correlation functions, we argue that the gauged WZW theory in the conformal limit is exactly the coset theory in CFT.

The examples given in Sec.5, can be rewritten as gauged WZW theories, in which case $G = G_L = G_R$. In the first example of real fermions, the non-linear sigma field ϕ is in the vector representation of G_L and the identity representation of G_R; in the second example, ϕ is in the adjoint representation of G_L and the identity representation of G_R. In both cases, the left gauge group is $H_L = G_L$ while H_R is absent. From this point of view, the chiral gauged WZW theory is more general than the vector gauged WZW theory.

In superstring theory, the Fadeev-Popov ghost system (and the related BRST operator) is obtained by gauge fixing the two-dimensional supergravity action which has a world-sheet local supersymmetry. To understand the ghost system in the fractional superstring case, it will be nice to find out the world-sheet local fractional supersymmetry from a two-dimensional action which involves the parafermion. This requires, as a first step, a classical action for the parafermion, which can be provided by the chiral gauged WZW theory. It is very similar to anyons in $2+1$ dimensions where a classical action for anyons requires the introduction of the Chern-Simon gauge coupling into the action for particles with integer spin or half-integer spin.

Finally the analogy between the gauged WZW theory and Polyakov's bosonic string theory as summarized in Table 2 is striking.

	Gauged WZW Theory	**Polyakov's String Theory**
Gauge Field	*Gauge connections*	*Two dimensional metric*
Gauge Symmetry	$v_L(\bar{z})$ *and* $v_R(z)$	*Diffeomorphism* $\xi^z\ ,\xi^{\bar{z}}$
Ghost System	$c^a, b^a_z, a=1,2,...,dimH_L$	c^z, b_{zz}
	$\bar{c}^\alpha, \bar{b}^\alpha_{\bar{z}}, \alpha=1,2,...dim\ H_R$	$c^{\bar{z}}, b_{\bar{z}\bar{z}}$
	$dim(c)=0,\ dim(b)=1$	$dim(c)=-1,\ dim(b)=2$
Remaining Fields	$h_{\rm p}$ *and* $\widetilde{h}_{\rm p}$	*Liouville field,* $\phi(z,\bar{z})=\phi(z)+\bar{\phi}(\bar{z})$
BRST Operator	$Q^L_{BRST}=\oint c^a\left(J^a+J^a_h+J^a_{gh}\right)$	$Q_{BRST}=\oint c\left(T_m+T_\phi+T_{gh}\right)$
	$Q^R_{BRST}=\oint \bar{c}^\alpha\left(\bar{J}^\alpha+\bar{J}^\alpha_{\tilde{h}}+\bar{J}^\alpha_{gh}\right)$	$\bar{Q}_{BRST}=\oint \bar{c}\left(\bar{T}_m+\bar{T}_{\bar{\phi}}+\bar{T}_{gh}\right)$
Critical Value	$k=-2C_2(H_L)=-2C_2(H_R)$	$c_m=\bar{c}_m=26$
Matter Fields	$G^{\Lambda\bar{\Lambda}}_{\lambda\bar{\lambda}}$	Φ_{mn}
Dressed Fields	$\Phi^{\Lambda\bar{\Lambda}}_{\lambda\bar{\lambda}}=G^{\Lambda\bar{\Lambda}}_{\lambda\bar{\lambda}}F(h_{\rm p},\ \widetilde{h}_{\rm p})$	$\widetilde{\Phi}_{mn}=\Phi_{mn}F(\phi)$

Table 2. Similarities between the gauged WZW theory and Polyakov's bosonic string theory.

In Table 2, Φ_{mn} are primary fields in the unitary minimal models with

$$c=1-\frac{6}{p(p+1)}. \tag{6.1}$$

See Ref.[22] for details.

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Appendix A.

In this Appendix we review the propreties of the non-abelian anomaly in $2n$-dimensions, using the differential geometry approach. The chiral anomaly was first derived in Ref.[16] while the relation between the anomaly in LR-scheme and that in the

A-scheme was first clarified in Ref.[20]. For a more comprehensive review, see Ref.[25,26]. Our purpose of including this appendix is to clarify the notations and make the paper self-contained. This review is longer than needed to derive the non-abelian anomaly, because the formalism will be needed in Appendix B.

The expressions of the anomalies involve only gauge fields and gauge transformation parameters since matter fields have been integrated out. The effective action in $2n$-dimensions, $W(A^L, A^R)$, can be written as an integral in $2n+1$-dimensions[1]

$$W(A^L, A^R) = c_n \int_{B^{2n+1}} \omega^0_{2n+1}(A^L, A^R) \ , \tag{A.1}$$

where ω^0_{2n+1} is a $(2n+1)$-form. The superscript indicates the power of v_L and v_R. So ω^0_{2n+1} does not depend on v_L, v_R explicitly. We are interested in the space, S^{2n}, which is the boundary of B^{2n+1}. The consistency between Eq.(2.4) and (A.1) requires that

$$\mathbf{d}\omega^1_{2n}(A^L, A^R;\ v_L,\ v_R) = \delta_{v_L, v_R}\omega^0_{2n+1}(A^L, A^R) \ . \tag{A.2}$$

The relation between the non-abelian anomaly in $2n$-dimensions and the abelian anomaly $\Omega_{2n+2}(A^L, A^R)$ in $(2n+2)$-dimensions was first shown in Ref.[16]:

$$\begin{aligned} \Omega_{2n+2}(A^L, A^R) &= \mathbf{d}\omega^0_{2n+1}(A^L, A^R) \\ \Omega_{2n+2}(A^L, A^R) = \mathrm{Tr}\Big(F_R^{n+1} - F_L^{n+1}\Big) &= \int_{t_1}^{t_2} dt \frac{d}{dt} \mathrm{Tr}\left[F(t)^{n+1}\right] \end{aligned} \tag{A.3}$$

with the boundary condition specified by

$$A(t_1) = A^L \ , \quad A(t_2) = A^R \qquad \text{and} \quad F(t) \equiv \mathbf{d}A(t) + A(t)^2 \ . \tag{A.4}$$

(Again, all products are wedge products.) Exploiting properties of the differential form we can rewrite the second equation of (A.3) as

$$\begin{aligned} \Omega_{2n+2}(A^L, A^R) &= \int_{t_1}^{t_2} dt (n+1) \mathrm{Tr}\left(\dot{F}(t) F^n(t)\right) \\ &= (n+1) \int_{t_1}^{t_2} dt \ \mathrm{Tr}\left\{\mathbf{d}\dot{A}(t) F^n(t) + \dot{A}(t)[A(t), F^n(t)]\right\} \\ &= (n+1) \int_{t_1}^{t_2} dt \ \mathbf{d}\ \mathrm{Tr}\left\{\dot{A}(t) F^n(t)\right\} \ , \end{aligned} \tag{A.5}$$

where we have used

(i) Tr $(AB) = -$ Tr (BA) if A and B are both odd forms ;

(ii) Bianchi identity, i.e. $DF = 0 = \mathbf{d}F + [A, F]$.

Comparing Eq.(A.3) with (A.5) we get

$$\omega^0_{2n+1}(A^L, A^R, \gamma) = (n+1) \int_{t_1}^{t_2} dt \ \text{Tr} \ \left\{ \dot{A}(t) F^n(t) \right\} \ . \tag{A.6}$$

Here a possible path choice, γ, in the expression of ω^0_{2n+1} is introduced because all previous discussions up to Eq.(A.6) depend only on the end points of the t-integration. Different choices of paths can be related to different choices of renormalization conditions.

There are two renormalization conditions frequently used. One is called the A-scheme in which the vector gauge transformation is invariant while the axial vector gauge transformation is anomalous. The other is called the LR-scheme in which the anomaly, ω^1_{2n}, and the effect action, W, can be separated into the left and the right pieces independently. In the following we will specify the paths γ_A (for the A-scheme) and γ_{LR} (for the LR-scheme). Hence the explicit expressions of $\omega^0_{2n+1}(A^L, A^R)$ and ω^1_{2n} can be evaluated by using Eq.(A.6) and (A.2). First of all, γ_A is chosen as in Fig.1, which is parametrized by

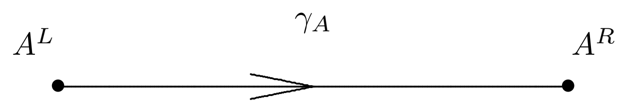

Fig. 1. The path γ_A is defined by the parametrization $A(t) \equiv V + tA, t \in [-1, 1]$ where $V \equiv (A^L + A^R)/2$ and $A \equiv (A^R - A^L)/2$.

$A(t) \equiv V + tA$, $t \in [-1, 1]$ where $V \equiv (A^L + A^R)/2$ and $A \equiv (A^R - A^L)/2$. With this path, γ_A, we can easily obtain

$$\omega^0_{2n+1}(A^L, A^R, \gamma_A) = (n+1) \int_{-1}^{1} dt \ \text{Tr} \ [\frac{1}{2}(A^R - A^L) F^n(t)] \ . \tag{A.7}$$

The vector gauge transformation is defined by $v \equiv v_L \equiv v_R$. And it is not hard to check the vector gauge invariance, *i.e.*,

$$\delta_{v,v} \omega^0_{2n+1}(A^L, A^R, \gamma_A) = 0 \ . \tag{A.8}$$

In two dimensions, Eq.(A.7) is equal to

$$\omega^0_3(A^L, A^R, \gamma_A) = \text{Tr} \ \left((A^R - A^L) \left[\mathbf{d}A^L + \mathbf{d}A^R \right] + \frac{2}{3}((A^R)^3 - (A^L)^3) \right) \ . \tag{A.9}$$

Then from Eq.(A.2) we can obtain

$$\omega^1_{2,A} = \text{Tr}\ \Big((\mathbf{d}v_R - \mathbf{d}v_L)(A^R + A^L) + (v_R - v_L)(A^R A^L + A^L A^R)\Big)\ . \tag{A.10}$$

On the other hand, the path γ_{LR} can be defined as in Fig.2,

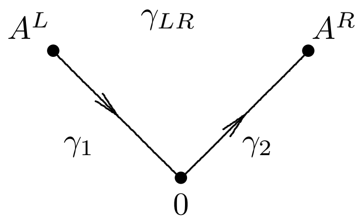

Fig. 2. The path γ_{LR} is parametrized by the following prescription:

along γ_1 : $A(t) = tA^L \qquad t \in [1, 0]$

along γ_2 : $A(t) = sA^R \qquad s \in [0, 1]$.

Then from Eq.(A.6) we can obtain $\omega^0_{2n+1}(A^L, A^R, \gamma_{LR})$ as

$$\begin{aligned} \omega^0_{2n+1}(A^L, A^R, \gamma_{LR}) &= (n+1)\left[\int_1^0 dt\ \text{Tr}\left(A^L F^n_L(t)\right) + \int_0^1 ds \text{Tr}\left(A^R F^n_R(s)\right)\right] \\ &\equiv \omega^0_{2n+1,C}(A^R) - \omega^0_{2n+1,C}(A^L) \end{aligned} \tag{A.11}$$

where $\omega^0_{2n+1,C}$ is the Chern-Simon form defined by

$$\begin{aligned} &\omega^0_{2n+1,C}(A) \equiv (n+1)\int_0^1 \text{Tr}\left(A \cdot F^n(t)\right) \\ &\text{and} \quad A(t) = tA\ , \quad F(t) = \mathbf{d}A(t) + A(t)^2\ . \end{aligned} \tag{A.12}$$

However, in two dimensions we have

$$\omega^0_{3,C}(A) = \text{Tr}\ \ (A\mathbf{d}A) + \frac{2}{3}\text{Tr}\left(A^3\right). \tag{A.13}$$

Thus $\omega^1_{2,LR}$ can be similarly evaluated from Eq.(A.2)

$$\begin{aligned} \mathbf{d}\omega^1_{2,LR}(A^L, A^R;\ v_L,\ v_R) &= \delta_{v_L,v_R}\omega^0_3(A^L, A^R, \gamma_{LR}) \\ = \delta_{v_R}\omega^0_{3,C}(A^R) - \delta_{v_L}\omega^0_{3,C}(A^L) &= \mathbf{d}\ \text{Tr}\left(\mathbf{d}v_R \cdot A^R\right) - \mathbf{d}\ \text{Tr}\left(\mathbf{d}v_L \cdot A^L\right) \end{aligned} \tag{A.14}$$

Therefore,

$$\omega^1_{2,LR}(A^L, A^R;\ v_L,\ v_R) = \text{Tr}\ (\mathbf{d}v_R\ A^R) - \text{Tr}\ (\mathbf{d}v_L\ A^L)\ . \tag{A.15}$$

which is Eq.(2.20).

From Fig.1 and Fig.2, it is clear that the connection between the A-scheme and the LR-scheme can be established through a closed-contour integral. To show this we first define the integral

$$\mathbf{d}\rho_{2n}(A^{(0)}, A^{(1)}, A^{(2)}) = (n+1)\oint_{\mathcal{C}} d\tau \mathrm{Tr}\left(\dot{A}(\tau)F^n(\tau)\right) \tag{A.16}$$

where $A(s,t) = A^{(0)} + sA^{(1)} + tA^{(2)}$ and $F(s,t) = \mathbf{d}A(s,t) + A^{(2)}(s,t)$. The contour $\mathcal{C}$ in the above integral is specified in Fig.3.

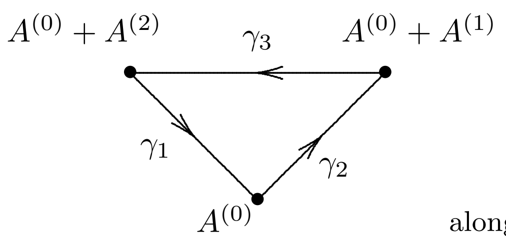

Fig. 3. The contour $\mathcal{C}$ in Eq.(A.16) is parametrized by the following prescription:

along $\gamma_1 : A(t) = A^{(0)} + tA^{(2)}$ $\quad t \in [1, 0]$.
along $\gamma_2 : A(s) = A^{(0)} + sA^{(1)}$ $\quad s \in [0, 1]$.
along $\gamma_3 : A(u) = A^{(0)} + (1-u)A^{(1)} + uA^{(2)}$ $\quad u \in [0, 1]$.

Eq.(A.16) can be evaluated in the following way. First the integrand can be viewed as an inner-product, in the vector space spanned by $d\vec{s}$ and $d\vec{r}$.

$$\begin{aligned}\dot{A}(\tau)F^n(\tau) &= dsA^{(1)}F^n(s,t) + dtA^{(2)}F^n(s,t) \\ &= (ds,\ dt)\cdot\left(A^{(1)}F^n(s,t)\ , A^{(2)}F^n(s,t)\right) \equiv d\vec{s}\cdot\vec{F}\ .\end{aligned} \tag{A.17}$$

From the Stoke's theorem, the closed-contour integral can be transformed into an area integral

$$\begin{aligned}\oint d\vec{s}\cdot\vec{F} &= \int d\vec{a}\cdot\vec{\nabla}\times\vec{F} \\ &= \int_{area} da\left(\frac{\partial}{\partial s}\left[A^{(2)}F^n(s,t)\right] - \frac{\partial}{\partial t}\left[A^{(1)}F^n(s,t)\right]\right)\end{aligned} \tag{A.18}$$

where the contour specifies the boundary of the area. This leads to a straightforward calculation and gives

$$\begin{aligned}
\mathbf{d}\rho_{2n} = (n+1)\int_0^1 ds \int_0^{1-s} dt \sum_{p=0}^{n-1} \\
\mathrm{Tr}\Big\{ A^{(2)}F^p(s,t)\left[\mathbf{d}A^{(1)} + A^{(1)}A(s,t) + A(s,t)A^{(1)}\right] F^{n-1-p}(s,t) \\
- A^{(1)}F^{n-1-p}(s,t)\left[\mathbf{d}A^{(2)} + A^{(2)}A(s,t) + A(s,t)A^{(2)}\right] F^{p}(s,t)\Big\} \\
= (n+1)\int_0^1 ds \int_0^{1-s} dt \sum_{p=0}^{n-1} \mathrm{Tr}\Big\{ A^{(2)}F^p(s,t)\mathbf{d}A^{(1)}F^{n-1-p}(s,t) \\
- A^{(1)}F^{n-1-p}\mathbf{d}A^{(2)}F^p(s,t) + A^{(2)}F^p(s,t)A^{(1)}\left[A(s,t),\ F^{n-1-p}(s,t)\right] \\
+ A^{(2)}\left[F^p(s,t),\ A(s,t)\right]A^{(1)}F^{n-1-p}(s,t)\Big\} \\
= (n+1)\int_0^1 ds \int_0^{1-s} dt\ \mathbf{d}\ \sum_{p=0}^{n-1} \mathrm{Tr}\left[-A^{(2)}F^p(s,t)A^{(1)}F^{n-1-p}(s,t)\right] \\
\Rightarrow \rho_{2n}(A^{(0)},A^{(1)},A^{(2)}) = (n+1)\sum_{p=0}^{n-1}\int_0^1 ds \int_0^{1-s} dt\ \mathrm{Tr}\left(A^{(1)}F^p(s,t)A^{(2)}F^{n-1-p}(s,t)\right)
\end{aligned} \tag{A.19}$$

In two dimensions we have

$$\rho_2(A^{(0)},A^{(1)},A^{(2)}) = 2\int_0^1 ds \int_0^{1-s} dt\ \mathrm{Tr}\left(A^{(1)}A^{(2)}\right) = \mathrm{Tr}\left(A^{(1)}A^{(2)}\right) \tag{A.20}$$

Hence the difference between $\omega^0_{2n+1}(A^L,A^R,\gamma_A)$ and $\omega^0_{2n+1}(A^L,A^R,\gamma_{LR})$ can be obtained if we compare the contours γ_A and γ_{LR} and use Eq.(A.7) and (A.11)

$$\omega^0_{2n+1}(A^L,A^R,\gamma_{LR}) - \omega^0_{2n+1}(A^L,A^R,\gamma_A) = \mathbf{d}\rho_{2n}(0,A^R,A^L)\ . \tag{A.21}$$

Furthermore the differences in the effective action and the anomaly can be obtained from Eq.(A.1), (A.2) and (A.21)

$$\begin{aligned}
W_{LR}(A^L,A^R) - W_A(A^L,A^R) = c_n \int_{S^{2n}} \rho_{2n}(0,A^R,A^L)\ , \\
\omega^1_{2n,LR}(A^L,A^R;\ v_L,\ v_R) - \omega^1_{2n,A}(A^L,A^R;\ v_L,\ v_R) = \delta_{v_L,v_R}\ \rho_{2n}(0,A^R,A^L).
\end{aligned} \tag{A.22}$$

Appendix B.

Following Ref.[20], we explain in some detail how to solve

$$\delta_{v_L,v_R}\Gamma\left(A^L,A^R,\phi\right) = \frac{k}{8\pi}\int \omega^1_{2,LR}(A^L,A^R;\ v_L,\ v_R)\ . \tag{B.1}$$

Using the WZ consistency condition, Eq.(2.6), Eq.(B.1) can be integrated along the following path in the functional space of ϕ, A^L, A^R:

$$\begin{aligned} \phi(x,t) &= g(x,t)\phi(x), \qquad A^R(x,t) = A^R(x) \\ A^L(x,t) &= g(x,t)A^L(x)g^{-1}(x,t) - \mathbf{d}_x g(x,t)\cdot g^{-1}(x,t) \end{aligned} \tag{B.2}$$

where t is the parameter specifying the path and $g(x,t)$ satisfies $g(x,t=0)=\mathbf{1}$ and $g(x,t=1)=\phi(x)^{-1}$. The particular ingredient in this choice is that one end point is $\Gamma\left(A^L, A^R, \phi\right)$, which is what we want, and the other end point is $\Gamma\left({A'}^L, {A'}^R, \phi=1\right)$, which is easy to evaluate because $\phi=1$.

To be more specific, the infinitesimal variations along t can be evaluated as

$$\begin{aligned} \delta\phi(x,t) &= (\Delta t)\frac{dg}{dt}g^{-1}\phi(x,t) \\ \delta A^L(x,t) &= -(\Delta t)\mathbf{d}_x\left(\frac{dg}{dt}g^{-1}\right) + [(\Delta t)\frac{dg}{dt}g^{-1},\ A^L(x,t)] \\ \delta A^R(x,t) &= 0\ . \end{aligned} \tag{B.3}$$

Together with Eq.(2.3) and (2.11), we can define the infinitesimal gauge transformation parameter as $v_L(x) \equiv (\Delta t)\frac{dg}{dt}g^{-1}$ and $v_R(x)=0$. Then the change of $\Gamma\left(A^L(t), A^R(t), \phi(t)\right)$ along the path can be treated as a result from the gauge transformation specified by the $v_L(x)$ and $v_R(x)$ above. Therefore, combining with Eq.(B.1) we have

$$\begin{aligned} \delta\Gamma\left(A^L(t), A^R(t), \phi(t)\right) &= \frac{k\Delta t}{8\pi}\int \omega^1_{2,LR}(A^L(t), A^R(t); \frac{dg}{dt}g^{-1}, 0)\ , \\ \Rightarrow\quad \frac{d}{dt}\Gamma\left(A^L(t), A^R(t), \phi(t)\right) &= \frac{k}{8\pi}\int \omega^1_{2,LR}(A^L(t), A^R(t); \frac{dg}{dt}g^{-1}, 0)\ . \end{aligned} \tag{B.4}$$

The integration over dt from 0 to 1 gives

$$\Gamma(A^L, A^R, \phi) = \Gamma(\phi^{-1}A^L\phi + \phi^{-1}\mathbf{d}\phi, A^R, 1) + \frac{k}{8\pi}\int dt\int \omega^1_{2,C}(A^L(t), \frac{dg}{dt}g^{-1}) \tag{B.5}$$

where we have used Eq.(A.15) to substitute for $\omega^1_{2,LR}$. The right-hand side can be solved in the following steps. We first notice that if $\phi=1$ and $v_L = v_R = v$, then $\delta\phi=0$ from Eq.(2.11). Therefore the first term is a solution of

$$\begin{aligned} \delta_{v,v}\Gamma(A^L, A^R, \phi=1) &= \frac{k}{8\pi}\int \omega^1_{2,LR}\left(A^L, A^R; v, v\right) \\ &= \frac{k}{8\pi}\int \delta_{v,v}\rho_2(0, A^R, A^L) \end{aligned} \tag{B.6}$$

where we have used Eq.(A.22) and the property that $\omega^1_{2,A}(A^L,A^R;v,v)=0$, and ρ_2 is given in (A.20). Thus we have the solution for the first term on the right hand side,

$$\Gamma(\phi^{-1}A^L\phi+\phi^{-1}\mathbf{d}\phi,A^R,1)=\frac{k}{8\pi}\int \mathrm{Tr}\left(A^R\phi^{-1}A^L\phi+A^R\phi^{-1}\mathbf{d}\phi\right)\ . \tag{B.7}$$

The second term can be simplified by using Zumino's result[16],

$$\int dt\int_{S^2}\omega^1_{2,C}(A^L(t),\frac{dg}{dt}g^{-1})=\frac{-1}{3}\int_{B^3}\mathrm{Tr}\left(\varphi^{-1}\mathbf{d}\varphi\right)^3+\int_{S^2}\mathrm{Tr}\left(-\mathbf{d}\phi\phi^{-1}A^L\right) \tag{B.8}$$

Combining Eq.(B.5), (B.7) and (B.8) we obtain

$$\begin{aligned}\Gamma(A^L,A^R,\phi)=&\frac{-k}{24\pi}\int_{B^3}\mathrm{Tr}\left(\varphi^{-1}\mathbf{d}\varphi\right)^3\\&+\frac{k}{8\pi}\int_{S^2}\mathrm{Tr}\left(A^R\phi^{-1}\mathbf{d}\phi-\mathbf{d}\phi\phi^{-1}A^L+A^R\phi^{-1}A^L\phi\right)\end{aligned} \tag{B.9}$$

This is the solution we quoted in Eq.(2.21).

Appendix C.

In this appendix the evaluation of the determinant $det(\mathrm{D}^R_z)det(\mathrm{D}^L_{\bar z})$ appearing in Eq.(3.9) will be shown. This result was calculated in the A-scheme by many authors[6,27,28]. For our purpose it must be appropriately translated to the LR-scheme. In the A-scheme, the result is

$$\left(det\mathrm{D}^R_z\right)\left(det\mathrm{D}^L_{\bar z}\right)\Big|_{\mathrm{A-scheme}}=\exp\left[2C_2(H)I(h\widetilde{h})\right]\cdot\left(det\ \delta^{\alpha\beta}\partial_z\right)\left(det\ \delta^{\alpha\beta}\partial_{\bar z}\right) \tag{C.1}$$

where the $C_2(H)$ is the dual Coxeter number, defined by the structure constant of the gauge group, $\sum_{\gamma\delta}f^{\alpha\gamma\delta}f^{\beta\gamma\delta}=-C_2(H)\delta^{\alpha\beta}$. For $SU(N)$, $C_2(SU(N))=N$. The summation over the indices $\alpha,\beta=1,2,...,dim\ H$. The vector gauge invariance requires $H_L=H_R=H$. The appearance of the WZW action indicates the existence of the anomaly when we transform D_μ to ∂_μ. In Sec.3, we need the result in the LR-scheme in which the left gauge fields are independent of the right ones. Using Eq.(A.22), we transform the effective action from the A-scheme to the LR-scheme,

$$\begin{aligned}W_{LR}(A^L,A^R)&=W_A(A^L,A^R)+\frac{2C_2(H)}{4\pi}\int\mathrm{Tr}\left(A^RA^L\right)\\&=2C_2(H)\left[-I(h\widetilde{h})-\frac{1}{4\pi}\int d^2z\mathrm{Tr}\left(\partial_z\widetilde{h}\widetilde{h}^{-1}h^{-1}\partial_{\bar z}h\right)\right]\\&=2C_2(H)\left[-I(h)-I(\widetilde{h})\right]\end{aligned} \tag{C.2}$$

where Eq.(3.3) is used. Therefore, we have

$$
\begin{aligned}
\left(det\mathrm{D}_z^R\right)\left(det\mathrm{D}_{\bar z}^L\right)\Big|_{\mathrm{LR-scheme}} &= \exp\left[2C_2(H)I(\widetilde{h})\right]\left(det\ \delta^{\alpha\beta}\partial_z\right) \\
&\quad\cdot\ \exp\left[2C_2(H)I(h)\right]\left(det\ \delta^{\alpha\beta}\partial_{\bar z}\right).
\end{aligned}
\tag{C.3}
$$

Naturally it leads to

$$
\begin{aligned}
\left(det\mathrm{D}_z^R\right)\left(det\mathrm{D}_{\bar z}^L\right)\Big|_{\mathrm{LR-scheme}} &= \exp\left[2C_2(H_R)I(\widetilde{h})\right]\left(det\ \delta^{\alpha\beta}\partial_z\right) \\
&\quad\cdot\ \exp\left[2C_2(H_L)I(h)\right]\left(det\ \delta^{\alpha\beta}\partial_{\bar z}\right)
\end{aligned}
\tag{C.4}
$$

where $\alpha,\beta = 1,2,...,dim\ H_R$ for the right gauge fields and $\alpha,\beta = 1,2,...,dim\ H_L$ for the left gauge fields.

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